Rowmotion on Doppelgänger Pairs

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Table of Contents

Abstract

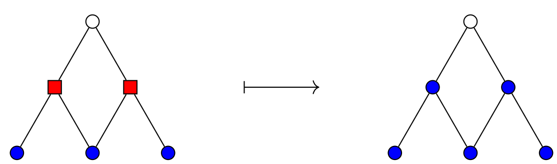
This thesis explores the action rowmotion in the combinatorial, piecewise-linear (PL), and birational settings. We focus on how doppelgängers with isomorphic comparability graphs behave under PL rowmotion. Hopkins [11] proved posets with isomorphic comparability graphs have a bijection between their orbits under rowmotion on order ideals that respects both the orbit lengths and the sum of the down-degrees across the orbit. He conjectures the analogous result should hold for PL rowmotion on P-partitions (Conjecture 4.38). Chapter 2 contains a modified proof of his proposition. It also confirms Hopkins' conjecture for two families: the broom and diamond posets. In the final chapter, we consider Hopkins' conjecture in terms of rowmotion on associated polytopes: the order, order-reversing, and chain polytopes. We do so to see if a natural bijection can be found across the polytopes that solves Hopkins' conjecture for any posets with isomorphic comparability graphs.

Introduction

A **p**artially **o**rdered **set** (*poset*) is a set with a binary relation, denoted $\leq$, that is reflexive, antisymmetric, and transitive. Let P be a finite poset. A downward closed subset $I \subseteq P$ is called an *order ideal*. The set of all such order ideals is denoted $J(P)$. This thesis will explore the action *rowmotion* on $J(P)$ and its generalizations, *piecewise-linear* (PL) rowmotion and *birational* rowmotion. In particular, we will examine when posets exhibit the same dynamics of PL rowmotion.

Rowmotion sends an order ideal $I \in J(P)$ to the order ideal generated by the minimal elements of $P \backslash I$.

Example 0.0.1. Below are Hasse diagrams for a poset P with an order ideal I shown by blue vertices in the image on the left. The red square vertices are the minimal elements of $P \backslash I$. Rowmotion sends I to the order ideal generated by those minimal elements, shown on the right.



Rowmotion has been studied in many contexts including as a permutation of antichains ([2], [8], [12]), as a permutation of monotone Boolean functions ([3]), and as a permutation of nonnesting partitions (antichains in the root poset of a finite Weyl group) ([16], [1]). In [3], Cameron and Fon-der-Flaass defined a group of permutations on monotone Boolean functions which we now call the *toggle group* and interpret as a permutation of $J(P)$. The toggle group consists of local involutions, called *toggles*, one corresponding to each element of P. The toggle t_v corresponding to $v \in P$ will add or remove v from an order ideal if the result is also an order ideal, or it will do nothing. Rowmotion is the composition of these toggles from top-to-bottom (reverse order of any linear extension). This alternate definition of rowmotion allows us to lift from the combinatorial setting to the piecewise-linear and then birational settings. PL rowmotion acts on functions on a poset P, specifically those functions within the *order polytope*, denoted $\mathcal{O}(P)$, (introduced by Stanley [19]). P-partitions, weakly order-preserving maps which partition a poset into $(\ell + 1)$ blocks by assigning the values $\{0, \ldots, \ell\}$, can be scaled by their height ℓ to sit within the rational points of the order polytope. In fact, every rational point can be considered a P-partition of height equal to the least common multiple of the denominators of the point's coordinates. Much study of PL rowmotion focuses on the restriction to these rational points. "Detropicalizing" the PL setting (replacing $(\max, +)$ with $(+, \cdot)$) takes us to the birational setting. Results proven for birational toggles and rowmotion are implied for the other settings. Rowmotion can also be studied on other polytopes associated with a poset P, including the *order-reversing polytope* $\mathcal{OR}(P)$ and the *chain polytope* $\mathscr{C}(P)$ [12].

Specific posets exhibit particularly nice behavior under rowmotion, making them of especial interest. Significant work has been done on the products of chains posets ([23], [17], [24]) and root posets ([16],[1], [11]), including showing the cyclic sieving phenomenon (introduced by [18]) as well as the homomesy phenomenon (introduced by [17]) for a number of relevant statistics. Studying these phenomena uncovered connections to binary words [23], noncrossing partitions [1], standard Young tableaux [23], and increasing tableaux [5].

In this thesis, the posets with nice behavior we will look at are *doppelgänger pairs*. Two posets are doppelgängers when they have equal *order polynomials*, or equivalently, they have the same number of P-partitions of height ℓ for $\ell \geq 1$. As it is not always easy to identify doppelgänger pairs, we will focus on posets with isomorphic *comparability graphs* which Stanley proved must be doppelgängers [19]. Hopkins proved there is a bijection between row-orbits of doppelgängers with isomorphic comparability graphs that preserves the orbit length and sum of the down-degrees across each orbit ([11], Proposition 4.10). He further conjectured that such a bijection could be found for the PL row-orbits of posets with isomorphic comparability graphs. This thesis expands work in support of that conjecture by proving it for what we call the *broom* and *diamond* poset families.

Chapter 1 begins with a brief review of the relevant poset theory. We then introduce the three settings of rowmotion—classical, piecewise-linear, and birational—showing how each subsequent lifting generalizes the last setting. We include connections to related objects: in Proposition 1.3.2 we define a bijection between row-orbits of the linear poset and binary necklaces, and in Section 1.5 we explore Galashin and Pylyavskyy's generalization of birational rowmotion: R-systems [9].

In the first section of Chapter 2, we provide a modified proof for the existence of a bijection between the row-orbits of posets with isomorphic comparability graphs that preserves the orbit length and sum of the down-degrees across each orbit, expanding the argument found in [11]. Section 2.2 reiterates Hopkins' conjecture of such a bijection existing in the piecewise-linear setting. Then in Proposition 2.2.3 we prove the conjecture for the doppelgänger infinite families we refer to as the diamond and broom posets.

Because posets with isomorphic comparability graphs must have isomorphic chain polytopes [19], in Chapter 3 we explore moving to rowmotion on the chain polytope hoping to find a method for proving the conjecture in Section 2.2 for generic doppelgängers. We utilize mappings between the poset's associated polytopes from the works of Stanley [19], Joseph [12], and Hopkins [11]. Example 3.0.15 bijects the order polytope of P to the order polytope of its doppelgänger Q by way of their chain polytopes: $\mathcal{O}(P) \to \mathscr{C}(P) \to \mathscr{C}(Q) \to \mathcal{O}(Q)$. The natural isomorphism used to move across the chain polytopes did not preserve the row-orbit structure, but it exhibited a pattern within the mixed row-orbits. In the future, this pattern could be disentangled to construct an alternative bijection between the chain polytopes with the desired properties.

Chapter 1

Rowmotion

1.1 Classical Rowmotion

We assume the reader is familiar with the basic theory of *partially ordered sets* (*posets*). In this section we will give relevant definitions and notation; however for a more in-depth introduction to poset theory, see [20]. Within this work **all posets will be finite** and represented using their Hasse diagrams.

Definition 1.1.1 ([20], Definition 4.1)**.** A **poset** P is a set with a binary relation denoted $\leq$ satisfying the following axioms:
(P1) (reflexivity) $v \leq v$ for all $v \in P$.
(P2) (antisymmetry) If $v \leq w$ and $w \leq v$, then $v = w$.
(P3) (transitivity) If $v \leq w$ and $w \leq x$, then $v \leq x$.

For $v, w \in P$, we say v *covers* w, denoted $w \lessdot v$ or $v \gtrdot w$, if $w < v$ and $\nexists\, x \in P$ such that $w < x < v$. We will use the following notation from [3]:
for $v \in P$,

$$\begin{aligned} \downarrow v &= \{w \in P : v \text{ covers } w\}; \\ \uparrow v &= \{w \in P : w \text{ covers } v\}. \end{aligned}$$

If either $v \leq w$ or $w \leq v$, they are said to be *comparable*, otherwise we say v and w are *incomparable*. An element v is called *maximal* (resp. *minimal*) if there is no $x \in P$ such that $x \gtrdot v$ (resp. $x \lessdot v$). Note that if v is minimal in P, then $\downarrow v$ is the empty set. Similarly if v is maximal in P, we find $\uparrow v$ is the empty set. We will denote the sets of maximal and minimal elements of P as $\max(P)$ and $\min(P)$ repsectively, distinguished from the customary max and min functions by the type of input.

A finite *chain* C of a poset P is a totally ordered subset of the form $x_0 < x_1 < \cdots < x_k$. A chain is *maximal* if it is not contained in any larger chain. Conversely, an *antichain* A of P is a subset which is pairwise incomparable.

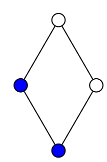
Definition 1.1.2. An *order ideal* (or *down-set*) of a poset P is a subset $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$.

Note both the empty set and P are order ideals. We denote the set of all order ideals of a poset P as $J(P)$. A subset S satisfying the flipped condition— if $x \in S$ and $x \leq y \in P$, then $y \in S$— is called a *filter* (or *up-set*) of P.

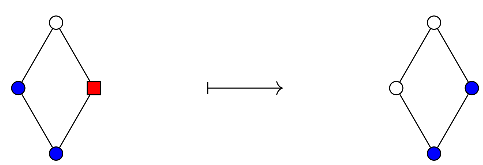
Definition 1.1.3. Let P be a finite poset. **Classical rowmotion** is the map $\mathrm{row}\colon J(P) \to J(P)$ which sends every order ideal I to the order ideal generated by the minimal elements of $P \backslash I$.

Classical rowmotion can be thought of as the composition of three bijective maps: 1) complementation, 2) map to the antichain of minimal elements, and 3) generation of the order ideal. For a more extensive break down of various types of rowmotion similarly defined via composition, such as rowmotion on filters and antichains, see [12].

Example 1.1.4. Let $P = [2] \times [2]$, and let I be the order ideal represented by the blue vertices in the Hasse diagram of P below.



To perform rowmotion on I, we first identify the red square vertex in the below Hasse diagram as the only minimal element in $P \backslash I$. Then $\mathrm{row}(I)$ is the order ideal generated by the minimal element (i.e., the minimal element and everything less than it) as shown on the right.



Cameron and Fon-der-Flaass [3] showed rowmotion on order ideals could be characterized by simple transformations. These transformations were later named *toggles* by Striker and Williams [23].

Definition 1.1.5. Let P be a poset and $v \in P$. Then a v-**toggle** is the permutation $t_v\colon J(P) \to J(P)$ defined by

$$t_v(I) = \begin{cases} I \cup \{v\}, & v \notin I \text{ but } \downarrow v \subseteq I, \\ I \backslash \{v\}, & v \in I \text{ but } \uparrow v \cap I = \emptyset, \\ I, & \text{otherwise.} \end{cases}$$

Note that each toggle is an involution. The alternate definition of rowmotion consists of a composition of toggles. For the composition of toggles we will use the following convention $t_v \circ t_w(I) = t_v t_w(I) = t_v(t_w(I))$.

Proposition 1.1.6. Two toggles, t_v and t_w, commute if and only if neither v nor w covers the other.

Proof. Let $v, w \in P$. Consider the following cases:

Case 1*:* Let $v = w$. Then $t_v \circ t_w = t_v \circ t_v = t_w \circ t_v$.

Case 2*:* Let $v < w$ or $w < v$ such that neither covers the other. Without loss of generality, assume $v < w$. Accordingly, there exists $x \in P$ such that $v < x < w$. For an order ideal I, $w \in I$ implies $v, x \in I$, and $v \notin I$ implies $x, w \notin I$. For this reason, either exactly one of the toggles, t_v or t_w, changes the order ideal or neither do, since we cannot change both the status of v and of w without also changing the status of x. In the case that neither change I, clearly $t_v \circ t_w(I) = I = t_v \circ t_w(I)$. Now, suppose t_v changes the order ideal, i.e., $t_v(I) \neq I$. Then $t_w(I) = I$ and $t_w \circ t_v(I) = t_v(I)$, since the status of x being unchanged by t_v prevents the status of w from changing in $t_w(t_v(I))$. Thus $t_w \circ t_v(I) = t_v(I) = t_v \circ t_w(I)$. The argument for if t_w changes the order ideal is similar.

Case 3*:* Let v and w be incomparable. Since v and w are incomparable, the inclusion or exclusion of w in I has no effect on the relation of $\downarrow v$ or $\uparrow v$ to I and thus no effect on the inclusion or exclusion of v in I when applying t_v. The same holds for the impact of v's status on the inclusion or exclusion of w in I under t_w. So $t_v \circ t_w = t_w \circ t_v$.

Case 4*:* Let v cover w or w cover v. Without loss of generality, assume w covers v. Let I be an order ideal in which w is a maximal element and the only cover of v in I. Then $t_v \circ t_w(I) = I \backslash \{v, w\}$ and $t_w \circ t_v(I) = I \backslash \{w\}$. Thus $t_v \circ t_w \neq t_w \circ t_v$.

□

Before we can characterize rowmotion by toggles, we need to define a *linear extension.*

Definition 1.1.7. A **linear extension** of P is a totally ordered sequence of the elements of P, $v_1 < v_2 < \cdots < v_n$, in which each element appears once and the partial order on P is preserved. In other words, if $v_i < v_j$ in P, then $v_i < v_j$ in the linear extension.

Given a linear extension of P, $v_1 < \cdots < v_j < v_{j+1} < \cdots < v_n$, where v_j and v_{j+1} are incomparable elements, we can swap their positions and get another valid linear extention of P, $v_1 < \cdots < v_{j+1} < v_j < \cdots < v_n$. Therefore, by Proposition 1.1.6, we find that the result of toggling by the elements of P in the order (or reverse order) of a linear extension is independent of the choice of linear extension. We are now ready to define rowmotion as a composition of toggles.

Theorem 1.1.8 ([3])**.** *For any linear extension ($v_1 < v_2 < \cdots < v_n$) of P, and for any order ideal $I \in J(P)$, the following holds:*

$$\mathrm{row}(I) = t_{v_1} \circ t_{v_2} \circ \cdots \circ t_{v_n}(I).$$

Proof. Let $I' = \mathrm{row}(I)$. For $i = 0, \ldots, n$, let I_i be the subset of P defined by

$$\begin{aligned} \chi_{I_i}(v_j) &= \chi_{I'}(v_j) && \text{for } j > i \\ \chi_{I_i}(v_j) &= \chi_{I}(v_j) && \text{for } j \leq i \end{aligned}$$

where χ_I denotes the characteristic function for the set I (resp. I_i, I'). It is enough to prove that for all $i > 0$, $t_{v_i}(I_i) = I_{i-1}$.

First we will show by cases that each I_i is an order ideal. Let $v_k, v_l \in P$ such that v_l covers v_k, and let $v_l \in I_i$.

Case 1*:* Let $l \leq i$. Note $k < l$ by virtue of the linear extension, so $k \leq i$. By assumption, $v_l \in I_i$, so $1 = \chi_{I_i}(v_l) = \chi_I(v_l)$, and thus $v_l \in I$. Because I is an order ideal and v_l covers v_k, we have $v_k \in I$. Therefore $1 = \chi_I(v_k) = \chi_{I_i}(v_k)$, indicating $v_k \in I_i$. Hence I_i is an order ideal.

Case 2*:* Let $l > k > i$. Since $v_l \in I_i$, we have $1 = \chi_{I_i}(v_l) = \chi_{I'}(v_l)$, hence $v_l \in I'$. As a result of rowmotion, I' is an order ideal, so $v_l \in I'$ implies $v_k \in I'$. Therefore, $1 = \chi_{I'}(v_k) = \chi_{I_i}(v_k)$ and $v_k \in I_i$ as desired.

Case 3*:* Let $l > i$ and $k \leq i$. As above, $v_l \in I_i$ implies $v_l \in I'$. There are two possibilities: $v_l \in I$ and was not removed by rowmotion or $v_l \notin I$ and was added by rowmotion. In the former case, since I is an order ideal, $v_l \in I$ implies $v_k \in I$, so $v_k \in I_i$. In the latter, $v_l \notin I$ but $v_l \in I'$ implies $v_l \in \min(P \backslash I)$, thus $v_k \in I$ and consequently, $v_k \in I_i$.
Hence each I_i is an order ideal.

Since both I_i and I_{i-1} are order ideals and v_i is the only vertex which can change status going from I_i to I_{i-1}, we need only consider the case in which $w \in I_i$ for $w \in \downarrow v_i$ and $w \notin I_i$ for $w \in \uparrow v_i$. But in this case

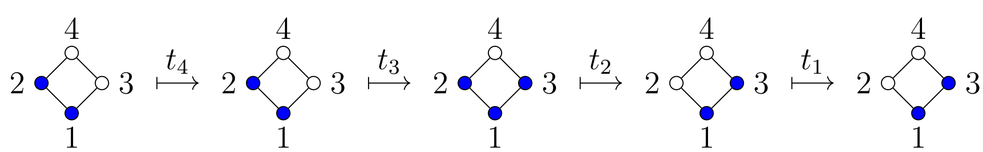
$$v_i \in I_i \iff v_i \in \max(I_i) \iff v_i \not< w, \text{ for all } w \in I' \iff v_i \notin I_{i-1}.$$

Further,

$$v_i \notin I_i \iff v_i \in \min(P \backslash I_i) \iff v_i \in \min(P \backslash I) \iff v_i \in I_{i-1}.$$

Thus $t_{v_i}(I_i) = I_{i-1}$. $\square$

Example 1.1.9. Let $P = [2] \times [2]$ and I be the order ideal represented by the blue vertices in the left Hasse diagram shown below. The vertex labels 1 through 4 are a linear extension on the poset P. By toggling in the order stated by Theorem 1.1.8, we get the same resulting order ideal as found when performing classical rowmotion in Example 1.1.4.



The above characterization of rowmotion as a composition of toggles reveals it is a toggle group action. Where the toggle group $G(P)$ is the subgroup of the symmetric group $\mathcal{G}_{J(P)}$ generated by $\{t_p\}_{p \in P}$. Further study has explored other group actions such as promotion [23], as well as generalized the toggle group to other settings which include chains, antichains, and independence sets [22]. Here we will briefly lay out a few conclussions surrounding the toggle group on order ideals.

Theorem 1.1.10 ([3], Theorem 4)**.** *Let P be a finite poset with a connected diagram, then the toggle group*

$$G(P) = \langle t_p \mid p \in P \rangle$$

contains the alternating group $A_{|J(P)|}$. If $P_1, \dots, P_k$ are the connected components of the diagram of P, then $G(P) = G(P_1) \times \cdots \times G(P_k)$.

Cameron and Fon-der-Flaass were able to conclude the toggle group of a connected poset P is symmetric or alternating, but did not find simple criterion for making the determination [3]. They did however prove the following relation on the toggles of adjacent poset elements.

Proposition 1.1.11 ([3], Lemma 5)**.** *Let $p, q \in P$ such that $p \lessdot q$. Then the permutation $t_p t_q$ has order 3 or 6. The order is 3 if and only if q is the only cover of p and p is the only element covered by q.*

Proof. There are three conditions for order ideals $I \in J(P)$ in relation to p and q: $p, q \notin I$, $p \in I$ but $q \notin I$, and $p, q \in I$. The permutation $\sigma = t_p t_q$ can only affect the status p and q with respect to an order ideal, so it is a permutation on these three states. Hence orbits of σ are of lengths 1, 2, or 3. Therefore the order of σ is 1, 2, 3, or 6.

For order ideals I such that q is the unique maximal element of I, we can see I has an orbit of length 3, $(I, I \backslash \{p, q\}, I \backslash \{q\})$, so $|\sigma| = 3$ or 6.

If there exists $r \in P$ such that $r \gtrdot p$ and $r \neq q$, then an order ideal I with $\max(I) = \{r, q\}$ has an orbit of length 2, $(I, I \backslash \{q\})$. Similarly, if there exists $t \in P$ such that $t \lessdot q$ and $t \neq p$, then an order ideal I such that $\min(P \backslash I) = \{t, p\}$ has an orbit of length 2, $(I, I \cup \{p\})$. In both of these cases, $|\sigma| = 3 \cdot 2 = 6$.

If no such r or t exist, then an order ideal I will either have a trivial orbit of length 1 or an orbit of length 3. Thus $|\sigma| = 3$. $\square$

1.2 Piecewise-linear Rowmotion

Another type of rowmotion, *piecewise-linear (PL) rowmotion*, generalizes rowmotion to the context of real-valued functions on a poset. The set of functions on which it is performed is called the *order polytope*. In defining the order polytope, we will need the following notation: for a poset P of size n, the n-dimensional real vector space of all functions $f : P \to \mathbb{R}$ is denoted $\mathbb{R}^P$.

Definition 1.2.1. The **order polytope** $\mathcal{O}(P)$ of a poset P is the subset of $\mathbb{R}^P$ defined by the following inequalities:

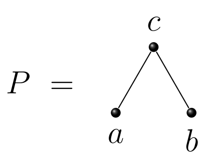
$$\begin{aligned} 0 \le f(x) \le 1 & \qquad \text{for all } x \in P \\ f(x) \le f(y) & \qquad \text{for } x \le y \text{ in } P. \end{aligned}$$

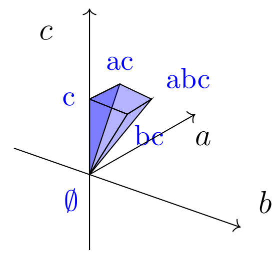
Note that $\mathcal{O}(P)$ is an n-dimensional convex polytope since it is bounded and defined by linear inequalities.

The following theorem is useful for visual depictions of the order polytope (see Example 1.2.3).

Theorem 1.2.2 ([19], Corollary 1.3)**.** *The vertices of $\mathcal{O}(P)$ are the characteristic functions χ_S of filters S of P. In particular, the number of vertices of $\mathcal{O}(P)$ is the number of filters of P.*

Example 1.2.3. For the poset P shown below, the order polytope $\mathcal{O}(P)$ is the set of all functions f such that $0 \leq f(a) \leq f(c) \leq 1$ and $0 \leq f(b) \leq f(c) \leq 1$. In the depiction of the order polytope on the right, each point has the form $(f(a), f(b), f(c))$ for some function satisfying the above conditions. The vertices are the characteristic functions of the filters of P and are labelled by those filters.





The geometric structure of the order polytope $\mathcal{O}(P)$ and the combinatorial structure of P are closely tied. For instance, the Ehrhart polynomial $i(\mathcal{O}(P), \ell)$ and the *order polynomial* $\Omega_P(\ell+1)$, both defined below, are equal ([19], Thm 4.1)[1]. Recall that for a d-dimensional convex polytope $\mathcal{P}$ in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, the Ehrhart polynomial $i(\mathcal{P}, m)$ gives the number of lattice points in the polytope $m\mathcal{P}$ with $m \in \mathbb{N}$. (A lattice point is a point with integer coordinate values). Additionally, if $n = d$, the leading coefficient of the Ehrhart polynomial equals the volume of $\mathcal{P}$ ([19], Section 4).

Definition 1.2.4. Let P be a poset of size n and let ℓ be a positive integer. Define $\Omega_P(\ell)$ to be the number of order-preserving maps $f\colon P \to \{1, \ldots, \ell\}$. Then the polynomial function of ℓ of degree n, $\Omega_P(\ell)$, is the **order polynomial of** P.

The leading coefficient of the order polynomial $\Omega_P(\ell)$ is equal to

$$\frac{\#\text{ of linear extensions of } P}{n!}.$$

Given the relation between the order polynomial and the Ehrhart polynomial, this is also the leading coefficient of $i(\mathcal{O}(P), \ell)$ and gives the volume of $\mathcal{O}(P)$.

Let $\hat{P}$ be the poset created from P by adjoining an extra minimal element, $\hat{0} < p$ for all $p \in P$, and an extra maximal element, $\hat{1} > p$ for all $p \in P$. All elements $f \in \mathcal{O}(P)$ can be thought of as elements in $\mathcal{O}(\hat{P})$ with the convention that $f(\hat{0}) = 0$ and $f(\hat{1}) = 1$.

[1] We follow the conventional indexing (as in [19]), however [11], from which we draw the conjecture that is the foundation of Chapter 2, would define the value of the order polynomial at ℓ to be our $\Omega_P(\ell+1)$.

Definition 1.2.5. For $v \in P$, the **piecewise-linear** v**-toggle** is the map ${t_v}^{PL} \colon \mathcal{O}(P) \to \mathcal{O}(P)$ (sometimes referred to as a *flip-map*) defined by

$$ {t_v}^{PL}(f)(p) = \begin{cases} f(p), & p \neq v, \\ \min\{f(x) : x \in \hat{P} \text{ covers } v\} + \max\{f(x) : v \text{ covers } x \in \hat{P}\} - f(v), & p = v. \end{cases} $$

Note that PL toggles are involutions and commute unless there is a covering relation between p and v. In fact, PL toggles generalize combinatorial toggles. For an order ideal $I \in J(P)$, consider the indicator function of its complement $\chi_{P\setminus I}$. We have ${t_v}^{PL}(\chi_{P\setminus I}) = \chi_{P\setminus t_v(I)}$ for all $I \in J(P)$ and $v \in P$.

While PL rowmotion can be defined as the composition of three maps, analogous to the three-map definition of classical rowmotion (as done by [12]), we will define PL rowmotion as a composition of toggles.

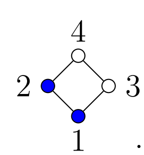
Definition 1.2.6. Let $f \in \mathcal{O}(P)$. We define **piecewise-linear (PL) rowmotion** to be the map $\text{row}^{PL} \colon \mathcal{O}(P) \to \mathcal{O}(P)$ such that

$$ \text{row}^{PL}(f) = {t_{v_1}}^{PL} \circ {t_{v_2}}^{PL} \circ \cdots \circ {t_{v_n}}^{PL}(f) $$

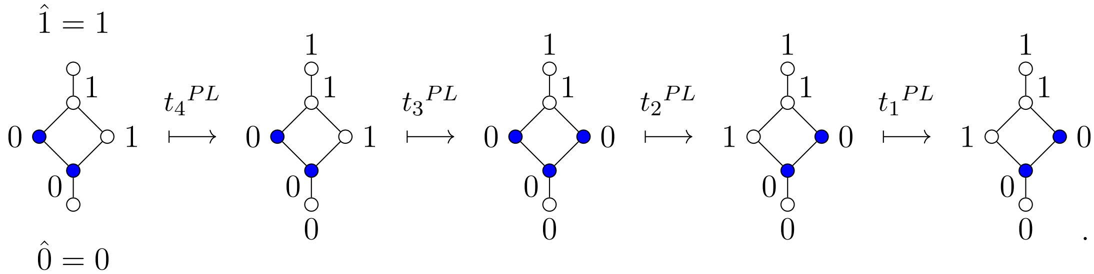
given any linear extension $(v_1 < v_2 < \cdots < v_n)$ of P.

When using this definition, it is clear that certain properties of PL toggles apply to PL rowmotion as well. For instance, PL toggles preserve the order polytope $\mathcal{O}(P)$, so PL rowmotion also preserves the order polytope $\mathcal{O}(P)$. Similarly, using the indicator functions which generalized combinatorial toggles to PL toggles, it follows that PL rowmotion is a generalization of classical rowmotion, as we will see in Example 1.2.7 below. Furthermore, since PL rowmotion is a generalization of classical rowmotion, we can consider classical rowmotion as acting on the order polytope. We will hold off on discussing rowmotion acting on other polytopes until Chapter 3, where we will examine the order-reversing polytope and the chain polytope.

Example 1.2.7. As in Example 1.1.9, let $P = [2] \times [2]$ and let I be the order ideal represented by the blue vertices shown below:



The vertex labels 1 through 4 denote the linear extension of P we will follow when toggling. On the Hasse diagrams for $\hat{P}$ below, with the labeling function $\chi_{P\setminus I} \in \mathcal{O}(P)$ extended in the conventional manner, we see that $\text{row}^{PL}(\chi_{P\setminus I}) = \chi_{P\setminus \text{row}(I)}$:



1.3 P-partitions

We can expand our view of rowmotion on the order polytope by scaling a class of functions called *P-partitions*.

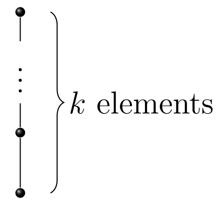
Definition 1.3.1. We define a **P-partition of height $\boldsymbol{\ell}$** to be a weakly order-preserving map $T\colon P \to \{0, 1, ..., \ell\}$. In other words, it is a function which partitions the poset P into $(\ell+1)$-blocks by assigning values $\{0, ..., \ell\}$ such that if $p \le q \in P$, then $T(p) \le T(q)$.

The set of P-partitions of height ℓ is denoted $PP^{\ell}(P)$. Recall the order polynomial of P that we saw in our exploration of piecewise-linear rowmotion. Since P-partitions are order-preserving maps from P to $\{0, 1, ..., \ell\} \simeq \{1, \ldots, \ell+1\}$, the number of P-partitions of height ℓ is given by $\Omega_P(\ell+1)$. Note P-partitions of height $\ell = 1$ are the same thing as order ideals, with elements in the order ideal having a value of 0 and those in the complement having a value of 1. Further, due to the weakly order-preserving property of P-partitions, each $I_i = T^{-1}(\{0, 1, \ldots, i\})$ is an order ideal for all $0 \le i \le \ell$. Additionally, P-partitions are $\mathbb{Z}$-labellings of P. Via the map $T \mapsto \frac{1}{\ell}T$, we find $PP^{\ell}(P)$ is in bijection with $\frac{1}{\ell}\mathbb{Z}^P \cap \mathcal{O}(P)$. In fact, every rational point in the order polytope corresponds to a P-partition of height equal to the least common multiple of the denominators of its coordinates. We already saw that piecewise-linear rowmotion preserves the order polytope. It is easy to check PL toggles, and subsequently PL rowmotion, preserve $\frac{1}{\ell}\mathbb{Z}^P$ for any $\ell \ge 1$. We can therefore pull back row^{PL} on $\frac{1}{\ell}\mathbb{Z}^P \cap \mathcal{O}(p)$, to act on $PP^{\ell}(P)$. We will abuse notation and also refer to this map as row^{PL}.

Below is a proposition relating the orbits of the linear poset under PL rowmotion to binary necklaces.

Proposition 1.3.2. *Let P be the linear poset of size k. Then the number of row-orbits for $PP^{\ell}(P)$, $\ell > 0$, equals the number of necklaces of $(k+1)$ black beads and ℓ white beads.*

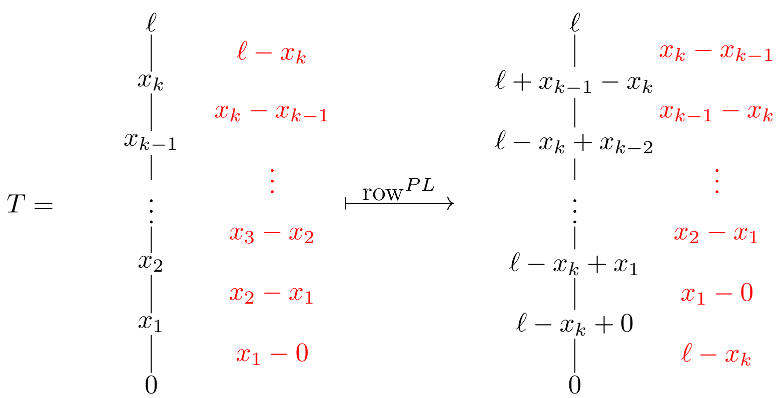
Proof. Let P be the linear poset of size k.

$P\ =$ 

There is a unique linear extension $x_1 < x_2 < \cdots < x_k$ of P. For $T \in PP^{\ell}(P)$, $0 \leq T(x_1) \leq T(x_2) \leq \cdots \leq T(x_k) \leq \ell$. When viewed on the extended poset $\hat{P}$, where $T(\hat{0}) = 0$ and $T(\hat{1}) = \ell$, the sum of the differences of adjacent elements in $\hat{P}$ must be equal to ℓ:

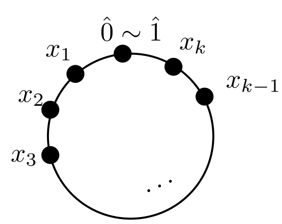
$$(\ell - T(x_k)) + (T(x_k) - T(x_{k-1})) + \cdots + (T(x_2) - T(x_1)) + (T(x_1) - 0)) = \ell - 0 = \ell.$$

Under PL rowmotion, the sequence of differences is simply cycled upwards by one position, with the difference between x_k and $\hat{1}$ coming around to be the difference between $\hat{0}$ and x_1. In the following diagram, each element is represented by its label in the P-partition. For the P-partition T, we use the labeling of the linear extension to represent the value T takes at that element of the poset, i.e., $x_i := T(x_i)$. The difference between adjacent elements is shown in red to the right of each P-partition below.



Therefore, we can view a row-orbit of $PP^{\ell}(P)$ as an equivalence class of sequences of differences that are equivalent under rotation.

Now we will construct a bijection between the row-orbits of $PP^{\ell}(P)$ and necklaces of $(k+1)$ black beads and ℓ white beads. First, we create a necklace of $(k+1)$ black beads out of the elements of $\hat{P}$ by letting $\hat{0} \sim \hat{1}$.



Next we choose a representative for the row-orbit, and using that sequence of differences, we place the ℓ white beads such that there are $(x_{i+1} - x_i)$ white beads between the black beads representing x_i and x_{i+1}. We can remove all labels on the beads since each possible labelling of the black beads is simply a different choice of the representative sequence for the row-orbit. To go from any necklace of $(k+1)$ black beads and ℓ white beads to the corresponding row-orbit we simply pick a black bead to be $\hat{0}$, which has $T(\hat{0}) = 0$, and as we move around the necklace we increase the value of T at the next element by the number of white beads between the black bead we currently are at and the next black bead. In other words, if we are at the black bead representing x_i with a value of $T(x_i)$, and there are 3 white beads and then a black bead, then $T(x_{i+1}) = T(x_i) + 3$. Since there is a bijection between row-orbits of $PP^\ell(P)$ and necklaces of $(k+1)$ black beads and ℓ white beads, there must be an equal number of each.

□

1.4 Birational Rowmotion

In this section we will examine a generalization of piecewise-linear rowmotion, birational rowmotion. While birational rowmotion is not the focus of this thesis, it is nonetheless useful to understand because the three settings (combinatorial (classical), piecewise-linear, and birational) are closely related, and insights proven at the birational level can be translated to the other settings.

Birational rowmotion and *birational toggling* as defined by Einstein and Propp [6, 7] draw from both the concepts of combinatorial toggles and toggles defined on Gelfand-Tsetlin patterns: the piecewise-linear version defined by Kirillov and Berenstein in [14], and the birational version appearing in Kirillov's work [13]. The birational setting is a "detropicalization" of the piecewise linear setting. The tropical operations $(\max, +)$ in PL toggles are replaced with the standard binary ring operations $(+, \times)$ (see [15]). The reader should note that rather than working over $\mathbb{R}_+$ as done by Einstein and Propp, we will instead work over fields as done by Grinberg and Roby in [10].

Definition 1.4.1 ([10])**.** Let P be a poset and $\mathbb{K}$ be a field. A $\mathbb{K}$-**labelling of P** is a map $f\colon \hat{P} \to \mathbb{K}$. The set of all $\mathbb{K}$-labellings of P is denoted $\mathbb{K}^{\hat{P}}$. If $f \in \mathbb{K}^{\hat{P}}$ and $v \in \hat{P}$, then $f(v)$ is called the *label of f at v*.

In the birational setting, the convention for extending a labeling of P to $\hat{P}$ is to define $f(\hat{0}) = f(\hat{1}) = 1$.

Definition 1.4.2. For $v \in P$, the **birational v-toggle** is the rational map $T_v\colon \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}}$ defined by

$$(T_v f)(p) = \begin{cases} \dfrac{\sum_{u \lessdot v} f(u)}{f(v)\sum_{u \gtrdot v} \frac{1}{f(u)}}, & p = v \\ f(p), & \text{otherwise.} \end{cases}$$

Note that T_v is an involution and a birational map.

Definition 1.4.3. We define **birational rowmotion** to be the rational map $\mathrm{row}^B\colon \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}}$ such that

$$\mathrm{row}^B(f) = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n}(f)$$

given any linear extension $(v_1 < v_2 < \cdots < v_n)$ of P.

In the next section, we will examine a generalization of birational rowmotion.

1.5 R-systems

This section looks at the work of Galashin and Pylyavskyy [9] in which they generalized birational rowmotion from posets to the class of arbitrary strongly connected directed graphs, resulting in what they named the *R-system*.

The R-system for a directed graph $G = (V, E)$ consists of iterating the map $X \mapsto X'$, where $X = (X_v)_{v \in V}$ and $X' = (X'_v)_{v \in V}$ are assignments of rational functions to the vertices of G which satisfy the toggle relations

$$X_v X'_v = \left(\sum_{(v,w) \in E} X_w \right) \left(\sum_{(u,v) \in E} \frac{1}{X'_u} \right)^{-1}, \text{ for all } v \in V \text{ [9]}.$$

Consider the edge $(u, v) \in E$ as corresponding to a cover relation $u \lessdot v \in P$. Comparing the above equation to the definition for birational toggles, it is noteable that the cover relations are flipped (i.e., the sums over $u \lessdot v$ in the toggle definition are over $v \lessdot w$ here) and the elements covered by v are already toggled. This implies a toggle relationship in the midst of rowmotion, not as an isolated mapping, and that rowmotion is being performed by toggling bottom to top (with flipped cover relations). In other words, birational rowmotion as utilized by Galashin and Pylyavskyy is equivalent to performing birational rowmotion on the dual of P, which we will denote P^d. For the above system to make sense and have solutions X' for generic X, G must be strongly connected.

There are a few definitions needed before we can precisely define R-systems.

Definition 1.5.1. Let $G = (V, E)$ be a directed simple graph (*digraph*). Then G is **strongly connected** if for any pair of vertices $u, v \in V$, there exists a directed path from u to v and a directed path from v to u.

Definition 1.5.2 ([9])**.** Let $\mathbb{S}$ be a ring and $\mathbb{K}$ its field of fractions. A **weighted digraph** $G = (V, E, \mathrm{wt})$ is a digraph with a **weight function** $\mathrm{wt}\colon E \to \mathbb{K}^*$ which takes non-zero values in $\mathbb{K}$. The canonical weight function assigns a weight of one to every edge.

Definition 1.5.3 ([9])**.** Let U be a non-empty finite set. Then the $(|U| - 1)$-**dimensional projective space over** $\mathbb{K}$, denoted $\mathbb{P}^U(\mathbb{K})$, is the set of all vectors $X = (X_u)_{u \in U} \in \mathbb{K}^U \backslash \{0\}$ modulo simultaneous rescalings by non-zero scalars $\lambda \in \mathbb{K}^*$.

Definition 1.5.4 ([9], Def 2.1)**.** Let G be a strongly connected digraph and $v \in V$. An **arborescence rooted at** v is a map $T\colon V\backslash\{v\} \to V$ such that

(1) for any $u \in V\backslash\{v\}$, we have $(u, T(u)) \in E$;

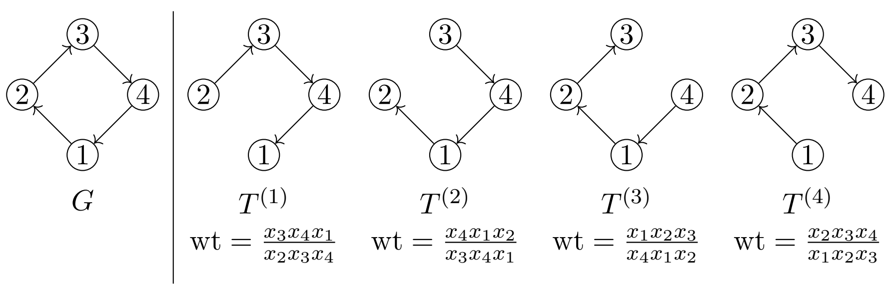
(2) for any $u \in V\backslash\{v\}$, there exists $k \in \mathbb{Z}_{>0}$ such that $T^k(u) = v$.

Another way to think of an arborescence at v is a collection of edges which form a spanning tree oriented towards v. The set of all arborescences rooted at v is denoted $\mathcal{T}(G, v)$.

Definition 1.5.5 ([9])**.** Given a point $X \in \mathbb{P}^V(\mathbb{K})$ with non-zero coordinates, the **weight of an arborescence** T, denoted $\mathrm{wt}(T; X) \in \mathbb{K}$, is defined as follows:

$$\mathrm{wt}(T; X) := \prod_{u \in V\backslash\{v\}} \mathrm{wt}(u, T(u)) \frac{X_{T(u)}}{X_u}.$$

Example 1.5.6. Shown on the left is a strongly connected digraph G with the canonical weight function $\mathrm{wt}(e) = 1$ for all $e \in E$. The canonical weight function places us in the *coefficient-free* R-system. Let $X = (x_1 : x_2 : x_3 : x_4)$. The graph G has four arborescences, shown with their weights on the right.



We will now introduce the system of equations that is the focus of this section. Let $G = (V, E, \mathrm{wt})$ be a weighted digraph and let $X = (X_v)_{v \in V}$ and $X' = (X'_v)_{v \in V}$. Consider the following system of equations ([9], Equation 2.1):

$$X_v X'_v = \left(\sum_{(v,w) \in E} \mathrm{wt}(v, w) X_w \right) \left(\sum_{(u,v) \in E} \frac{\mathrm{wt}(u, v)}{X'_u} \right)^{-1}, \text{ for all } v \in V. \tag{1.1}$$

Alternatively written ([9], Equation 2.2):

$$\sum_{(u,v) \in E} \mathrm{wt}(u, v) \frac{X'_v}{X'_u} = \sum_{(v,w) \in E} \mathrm{wt}(v, w) \frac{X_w}{X_v}, \text{ for all } v \in V. \tag{1.2}$$

Equation (1.1) can be considered a systems of equations on $\mathbb{P}^V(\mathbb{K})$, where X is the input and X' the output, since X and X' satisfy (1.1) if and only if λX and $\mu X'$ give a solution to (1.1): $\lambda, \mu \in \mathbb{K}^*$ and $\lambda X := (\lambda X_v)_{v \in V}$.

Theorem 1.5.7 ([9], Theorem 2.3)**.** *Let $G = (V, E, \mathrm{wt})$ be a strongly connected weighted digraph. Then there exists a birational map $\phi\colon \mathbb{P}^V(\mathbb{K}) \to \mathbb{P}^V(\mathbb{K})$ defined on some Zariski open subset $O \subset \mathbb{P}^V(\mathbb{K})$ such that for each $X \in O$, there exists a unique $X' \in \mathbb{P}^V(\mathbb{K})$ which gives a solution to (1.1), and we have $\phi(X) = X'$. A formula for $X' = (X'_v)_{v\in V}$ is given by*

$$X'_v = \frac{X_v}{\sum\limits_{T\in\mathcal{T}(G,v)} \mathrm{wt}(T;X)}. \tag{1.3}$$

While we have already given a loose definition of R-systems, now that we have precisely defined our mapping ϕ, let's return to the definition of the R-system.

Definition 1.5.8 ([9], Def 2.5)**.** Let $G = (V, E, \mathrm{wt})$ be a strongly connected weighted digraph. The **R-system** associated with G is a discrete dynamical system consisting of iterative application of the map ϕ. In other words, for $\mathcal{I} \in \mathbb{P}^V(\mathbb{K})$, the R-system is a family $(R(t))_{t\geq 0}$ of elements of $\mathbb{P}^V(\mathbb{K})$ defined as $R(t) = \phi^t(\mathcal{I})$ for $t \geq 0$.

Example 1.5.9. Returning to the graph in Example 1.5.6, we utilize Theorem 1.5.7 to find the solution X' to Eq (1.1):

$$\begin{aligned}
x'_1 &= \frac{x_1}{\mathrm{wt}(T^{(1)};X)} = \frac{x_1x_2x_3x_4}{x_1x_3x_4} = x_2;\\
x'_2 &= \frac{x_2}{\mathrm{wt}(T^{(2)};X)} = \frac{x_2x_1x_3x_4}{x_1x_2x_4} = x_3;\\
x'_3 &= \frac{x_3}{\mathrm{wt}(T^{(3)};X)} = \frac{x_3x_1x_2x_4}{x_1x_2x_3} = x_4;\\
x'_4 &= \frac{x_4}{\mathrm{wt}(T^{(4)};X)} = \frac{x_4x_1x_2x_3}{x_2x_3x_4} = x_1.
\end{aligned}$$

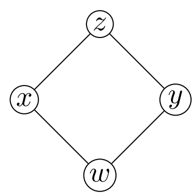
We now check that this indeed gives us a solution to Eq (1.1):

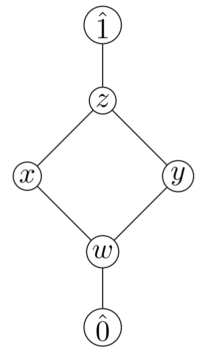
$$\begin{aligned}
x_1x'_1 &= (x_2)(\frac{1}{x'_4})^{-1} = x_2x'_4 = x_2x_1 = x'_1x_1\\
x_2x'_2 &= (x_3)(\frac{1}{x'_1})^{-1} = x_3x'_1 = x_3x_2 = x'_2x_2\\
x_3x'_3 &= (x_4)(\frac{1}{x'_2})^{-1} = x_4x'_2 = x_4x_3 = x'_3x_3\\
x_4x'_4 &= (x_1)(\frac{1}{x'_3})^{-1} = x_1x'_3 = x_1x_4 = x'_4x_4.
\end{aligned}$$

1.5.1 Generalization of Birational Rowmotion

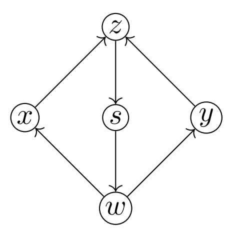
To see how an R-system is a generalization of birational rowmotion take a poset P and construct a digraph $G = G(P)$ from the extended poset $\hat{P}$ using the following method described in [9] Remark 2.7. First orient every edge of the Hasse diagram upwards, turning each covering relation $u \lessdot v$ into an edge (u, v). Then identify the vertices $\hat{0}$ and $\hat{1}$ as a single vertex s so that $V = P \cup \{s\}$. Let G have the canonical weight function assigning each edge a weight of 1.

Example 1.5.10. Let P and $\hat{P}$ be the following posets:

$P =$ 

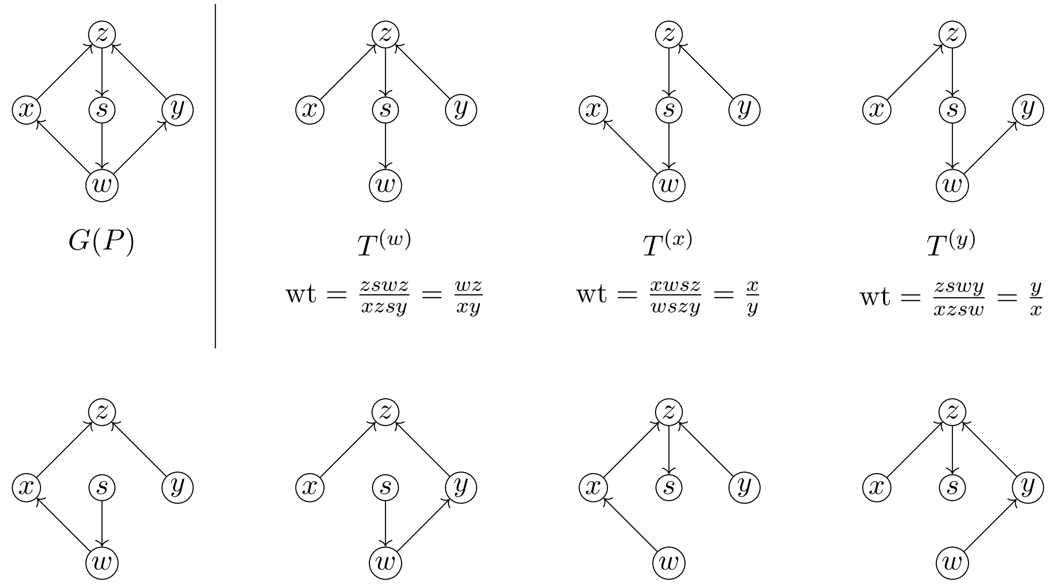
$\hat{P} =$ 

Then, following the construction method described above, we get the strongly connected digraph $G(P)$.

$G(P) =$ 

As noted by Galashin and Pylyavskyy in [9] Remark 2.7, one iteration of birational rowmotion on the dual of P, P^d, gives the unique solution to Eq (1.1) after removing the equation corresponding to s. By Theorem 1.5.7, a solution to the entire system exists, so it must coincide with the output of birational rowmotion on the dual poset. Due to the differing domains upon which the functions operate, $\mathbb{P}^V(\mathbb{K})$ vs $\mathbb{K}^P$, to reduce the system we must rescale the entries of $R(t)$ such that $R_s(t) = 1$ for all $t \geq 0$.

Example 1.5.11. Returning to the poset P and its associated digraph $G(P)$ from Example 1.5.10, let's check the output of the formula from Theorem 1.5.7 in fact coincides with the output of $\mathrm{row}^B(P^d)$ after rescaling. We start by finding the arborescences of $G(P)$ and their weights.



$G(P)$

$T^{(w)}$, $\mathrm{wt} = \frac{zswz}{xzsy} = \frac{wz}{xy}$ $\quad$ $T^{(x)}$, $\mathrm{wt} = \frac{xwsz}{wszy} = \frac{x}{y}$ $\quad$ $T^{(y)}$, $\mathrm{wt} = \frac{zswy}{xzsw} = \frac{y}{x}$

$T^{(z)}$, $\mathrm{wt} = \frac{xzzw}{wxys} = \frac{z^2}{ys}$ $\quad$ $T^{(z2)}$, $\mathrm{wt} = \frac{zzyw}{xyws} = \frac{z^2}{xs}$ $\quad$ $T^{(s)}$, $\mathrm{wt} = \frac{xzsz}{wxzy} = \frac{sz}{wy}$ $\quad$ $T^{(s2)}$, $\mathrm{wt} = \frac{zszy}{xzyw} = \frac{zs}{xw}$

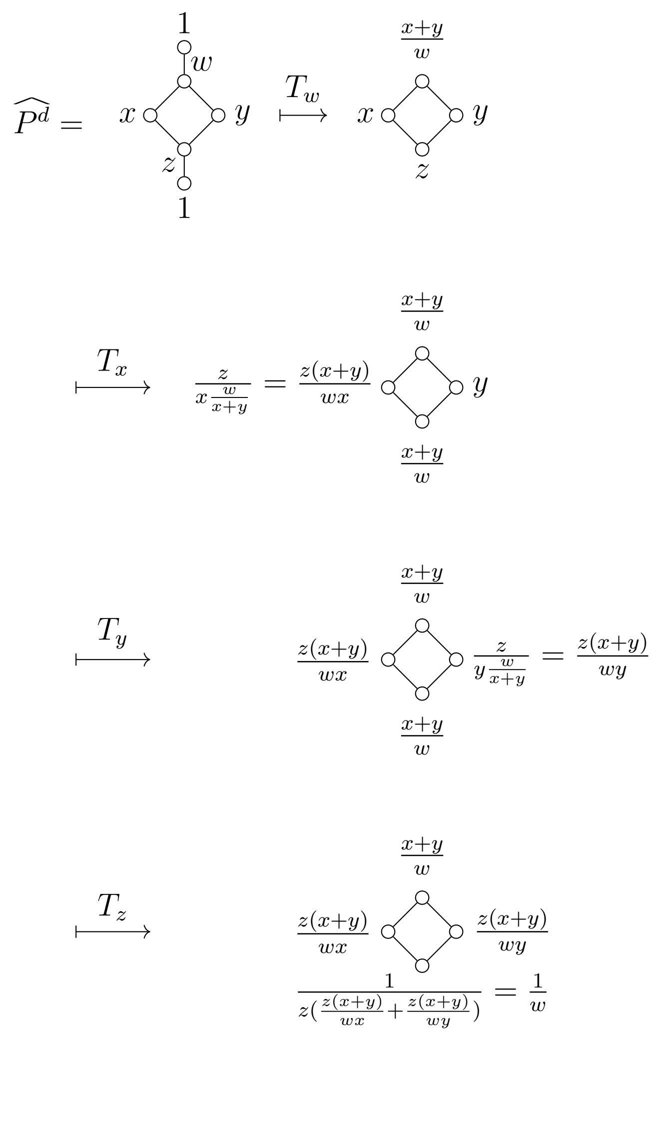
Then we use Equation 1.3 to calculate X'.

$$\begin{aligned}
w' &= \frac{w}{\mathrm{wt}(T^{(w)};X)} = \frac{w}{\frac{wz}{xy}} = \frac{xy}{z};\\
x' &= \frac{x}{\mathrm{wt}(T^{(x)};X)} = \frac{x}{\frac{x}{y}} = y;\\
y' &= \frac{y}{\mathrm{wt}(T^{(y)};X)} = \frac{y}{\frac{y}{x}} = x;\\
z' &= \frac{z}{\mathrm{wt}(T^{(z)};X) + \mathrm{wt}(T^{(z2)};X)} = \frac{z}{\frac{z^2}{ys} + \frac{z^2}{xs}} = \frac{sxy}{z(x+y)};\\
s' &= \frac{s}{\mathrm{wt}(T^{(s)};X) + \mathrm{wt}(T^{(s2)};X)} = \frac{s}{\frac{sz}{wy} + \frac{sz}{xw}} = \frac{wxy}{z(x+y)}.
\end{aligned}$$

We leave it to the reader to check this is a solution to Eq. (1.1). We now rescale the elements so that $R_s(t) = 1$ for all $t \geq 0$ ($s' = 1$ and $s = 1$).

$$w^* = \frac{xy}{z}\frac{z(x+y)}{wxy} = \frac{x+y}{w}; \qquad x^* = y\frac{z(x+y)}{wxy} = \frac{z(x+y)}{wx};$$
$$y^* = x\frac{z(x+y)}{wxy} = \frac{z(x+y)}{wy}; \qquad z^* = \frac{sxy}{z(x+y)}\frac{z(x+y)}{wxy} = \frac{s}{w} = \frac{1}{w}.$$

Let's now calculate $\mathrm{row}^B(P^d)$ to verify the equality of the outputs. We stop showing the elements $\hat{0}$ and $\hat{1}$ after the first depiction of $\widehat{P^d}$ for greater legibility. Recall these elements will always have a label value of 1.



As expected, the resulting values of $\mathrm{row}^B(P^d)$ match our rescaled outputs from the R-system.

The weighted Laplacian

Galashin and Pylyavskyy prove Theorem 1.5.7 in [9] Section 6 by cleverly rewriting the system of equations into a weighted Laplacian matrix of G and utilizing the Matrix-Tree theorem as we will show below.

Equation 1.1 can be rewritten yet again to be a linear system of equations in the variable $T := \left(\frac{X_u}{X'_u}\right)|_{u \in V}$ ([9], Eq (6.2)):

$$\sum_{(u,v)\in E} \mathrm{wt}(u,v)\frac{X_v}{X_u}\frac{X_u}{X'_u} = \sum_{(v,w)\in E} \mathrm{wt}(v,w)\frac{X_w}{X_v}\frac{X_v}{X'_v}, \quad \text{for all } v \in V. \tag{1.4}$$

The matrix $A = (a_{vu})$ representing this system is given by

$$a_{vu} = \begin{cases} \sum\limits_{(v,w)\in E} \mathrm{wt}(v,w)\frac{X_w}{X_v}, & \text{if } u = v; \\ -\mathrm{wt}(u,v)\frac{X_v}{X_u}, & \text{if } (u,v) \in E; \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

Then A is a weighted Laplacian matrix of G. The edge weights given by $\mathrm{wt}(u,v)\frac{X_v}{X_u}$. Thus its cokernel, the unique up to scalar multiples solution T of the system $AT = 0$, is given by the Matrix-Tree theorem, from which we get the formula in Theorem 1.5.7 ([9]). Recall the Matrix-Tree theorem tells us the determinant of the reduced Laplacian $\tilde{L}$ (the $(n-1)\times(n-1)$ created by removing the i-th row and column of L) is given by

$$\det \tilde{L} = \sum_{T \in \mathcal{T}(G,v)} \mathrm{wt}(T;X) \qquad\qquad \text{([4], Remark 9.6).}$$

This proof is an example of the connection between R-systems and sandpile theory (see [4]) described in [9] Remark 2.9. Galashin and Pylyavskyy observed that by setting $X_u = 1$ for all $u \in V$ and using the canonical weight function of every edge weight equal to 1, the denominator of the right hand side of 1.5.7 equals the size of the *critical group* (alternatively called the *sandpile group*) of G with a sink at v. Here they also noted that without setting $X_u = 1$, the formula was an expression of the cokernel of the weighted Laplacian as used above. The depth of the connection between sandpile theory and R-systems, and thus rowmotion, is a potential area of interest for future research.

Chapter 2

Rowmotion on a Special Class of Doppelgänger Pairs

This chapter will utilize classical rowmotion and PL rowmotion in the P-partition setting with a focus on *doppelgänger* posets. In defining *doppelgängers* we turn once again to the order polynomial.

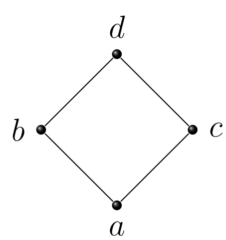
Definition 2.0.1. The posets P and Q are **doppelgängers** if $\Omega_P(\ell) = \Omega_Q(\ell)$.

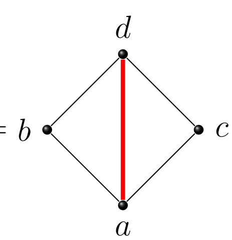
When looking for doppelgänger pairs a useful tool is their *comparability graphs*.

Definition 2.0.2. The **comparability graph** of a poset P, $\mathrm{com}(P)$, is the undirected, simple graph that connects each pair of vertices $p, q \in P$ if and only if p and q are comparable in P (i.e., $p \leq q$ or $q \leq p$).

A comparability graph can be thought of as the Hasse diagram with additional edges added to comparable elements that are not neighbors as seen below in Example 2.0.3.

Example 2.0.3. Consider the poset P. Note the elements a and d are comparable but neither is a cover for the other. In the comparability graph for P you can see an edge which didn't appear in the Hasse diagram that connects a and d, shown in red.

P =


$\mathrm{com}(P)$ =


Comparability graphs are not unique to a poset, i.e., it is not possible to find P from $\mathrm{com}(P)$. However, there are properties called *comparability invariants* which depend only on the comparability graph. One of these properties is the order polynomial, which leads us to how comparability graphs are used to find doppelgänger pairs.

Theorem 2.0.4 ([19]). *Let P and Q be posets with* $\mathrm{com}(P) \simeq \mathrm{com}(Q)$*. Then P and Q are doppelgängers.*

Note that not all doppelgängers need have isomorphic comparability graphs, however we will focus on doppelgängers which do have this property.

The following terms will help us to understand the condition that tells us when two posets will have isomorphic comparability graphs.

Definition 2.0.5. Let P be a poset. A subset A of P is **autonomous** if each $y \in P\backslash A$ has the same order relation to every element in A. In other words, for all $a, a' \in A$ and $y \in P\backslash A$,

$$(a \leq y \iff a' \leq y) \text{ and } (y \leq a \iff y \leq a').$$

Definition 2.0.6. Let $A \subseteq P$ be an autonomous subset. Then the poset Q is **obtained from P by dualizing A** when Q has the same elements as P and the same order relations except that those inside A are reversed:

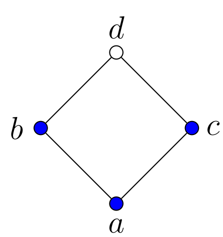
$$\begin{aligned} x \leq_Q y &\iff x \leq_P y && \text{for } x \in P\backslash A \text{ and } y \in P \\ x \leq_Q y &\iff x \geq_P y && \text{for } x, y \in A. \end{aligned}$$

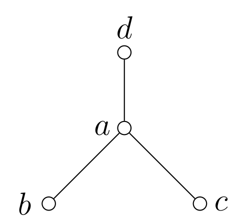
This brings us to the condition for isomorphic comparability graphs.

Lemma 2.0.7 ([11], Lemma 2.4). *The posets P and Q satisfy* $\mathrm{com}(P) \simeq \mathrm{com}(Q)$ *if and only if there is a sequence of posets $P = P_0, P_1, \ldots, P_k = Q$ such that P_i is obtained from P_{i-1} by dualizing an autonomous subset of P_{i-1} for $1 \leq i \leq k$.*

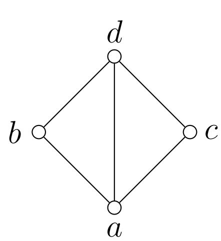
Let's look at an example of dualizing an autonomous set to find a doppelgänger pair.

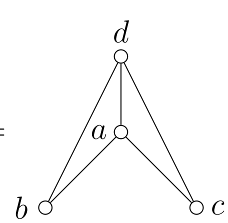
Example 2.0.8. Below we have a poset P with an autonomous subset $A = \{a, b, c\}$. After dualizing A we produce the poset Q.

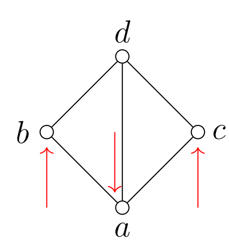
$P =$ 

$Q =$ 

Now let's look at their comparability graphs. With some slight adjustments to the depiction of $\mathrm{com}(Q)$, without changing any of its structure, we can easily see it is isomorphic to $\mathrm{com}(P)$, comfirming that P and Q are doppelgängers.

$\mathrm{com}(P) =$ 

$\mathrm{com}(Q) =$ 

$=$ 

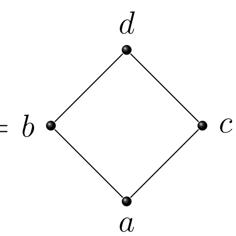
In our study of doppelgängers we will be utilizing the *down-degrees* of elements, order ideals, and P-partitions.

Definition 2.0.9. The **down-degree of an element** $p \in P$, denoted $\mathrm{ddeg}(p)$, is the number of elements p covers. See Example 2.0.10 below.
The **down-degree of an order ideal** I, $\mathrm{ddeg}(I)$, is the number of elements I covers within the poset of $J(P)$ ordered by inclusion. This gives us $\mathrm{ddeg}(I) = \#\max(I)$, i.e., the down-degree of an order ideal I is equal to the number of maximal elements in I. See Example 2.0.11 for the down-degrees of order ideals across a row-orbit.
The **down-degree of a P-partition** T of height ℓ is defined to be

$$\mathrm{ddeg}(T) := \sum_{i=0}^{\ell-1} \mathrm{ddeg}(T^{-1}\{0,1,\dots,i\}).$$

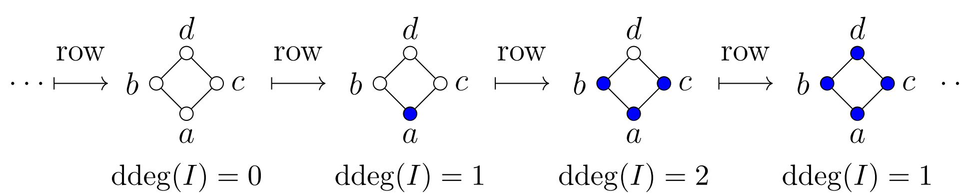
Each $T^{-1}\{0,1,\dots,i\}$ is an order ideal whose down-degree is calculated as previously defined. Example 2.0.12 below shows a calculation of the down-degree of a P-partition.

Example 2.0.10. Consider the poset P below.

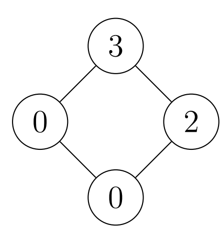
$P =$ 

In this case, $\mathrm{ddeg}(a) = 0$, $\mathrm{ddeg}(b) = 1 = \mathrm{ddeg}(c)$, $\mathrm{ddeg}(d) = 2$.

Example 2.0.11. For the following row_P-orbit, below each order ideal, shown in blue, is the corresponding down-degree.



Example 2.0.12. For the below P-partition, $\mathrm{ddeg}(T) = 1 + 1 + 2 = 4$.

$T \in PP^3(P)$ 

We will also be utilizing *toggleability statistics* which indicate whether an element can be toggled into or out of an order ideal. For each element $p \in P$ we define the toggleability statistics $\mathcal{T}_{p^+}, \mathcal{T}_{p^-}, \mathcal{T}_p \colon J(P) \to \mathbb{Z}$ as follows:

$$\mathcal{T}_{p^+}(I) := \begin{cases} 1 & \text{if } I \subsetneq t_p(I) \text{ (i.e., } p \text{ is minimal in } P \backslash I); \\ 0 & \text{otherwise;} \end{cases}$$
$$\mathcal{T}_{p^-}(I) := \begin{cases} 1 & \text{if } t_p(I) \subsetneq I \text{ (i.e., } p \text{ is maximal in } I); \\ 0 & \text{otherwise;} \end{cases}$$
$$\mathcal{T}_p(I) := \mathcal{T}_{p^+}(I) - \mathcal{T}_{p^-}(I).$$

Toggleability statistics are useful in a number of ways. For instance, they give us another way to write the down-degree of an order ideal. Specifically, $\operatorname{ddeg}(I) = \sum_{p \in P} \mathcal{T}_{p^-}(I)$ for all $I \in J(P)$. Additionally, as observed by [21], $\mathcal{T}_p$ is *homomesic* with respect to rowmotion as explored in the next lemma. A statistic on a set of objects is *homomesic* with respect to a bijective action if the average of the statistic over each orbit $\mathcal{O}$ is the same for all $\mathcal{O}$, i.e., the average does not depend on the choice of $\mathcal{O}$.

Lemma 2.0.13. *Given any $p \in P$, $\mathcal{T}_p$ is homomesic with respect to rowmotion with an average value of* 0.

Proof. Let $p \in P$ and $\mathcal{O} \subseteq J(P)$ be a row-orbit. As we traverse the row-orbit, if p is minimal in the complement $P \backslash I$ giving $\mathcal{T}_{p^+}(I) = 1$ then it will be maximal in $\operatorname{row}(I)$, so $\mathcal{T}_{p^-}(\operatorname{row}(I)) = 1$ and $\mathcal{T}_p(I) + \mathcal{T}_p(\operatorname{row}(I)) = 0$. Similarly, if p is maximal in I, p is minimal in $P \backslash \operatorname{row}^{-1}(I)$, so $\mathcal{T}_p(I) + \mathcal{T}_p(\operatorname{row}^{-1}(I)) = 0$. If p is not maximal in I nor minimal in the complement, $\mathcal{T}_p(I) = 0$. Thus $\sum_{S \in \mathcal{O}} \mathcal{T}_p(I) = 0$. □

2.1 Rowmotion on Order Ideals

This section will focus on how posets with isomorphic comparability graphs behave the same way under rowmotion, giving a modified proof of Hopkins' Proposition 4.10 ([11]). To distinguish between the various rowmotion operators, we will follow the example of Hopkins and use the notation $\operatorname{row}_P \colon J(P) \to J(P)$ to denote classical rowmotion on the order ideals of P.

To start, let P and Q be posets with $\operatorname{com}(P) \simeq \operatorname{com}(Q)$. Then by Lemma 2.0.7, we can assume Q is obtained from P by dualizing a non-empty autonomous subset $A \subseteq P$. We will also use A to denote the subposet of P formed by the elements of A and call the induced subposet formed by the elements of A in Q, A^*. Define the subsets $U, L, N \subseteq P$ as follows

$$U := \{p \in P : p > a \text{ for all } a \in A\};$$
$$L := \{p \in P : p < a \text{ for all } a \in A\};$$
$$N := \{p \in P : p \text{ is incomparable to } a \text{ for all } a \in A\}.$$

Note that since A is autonomous, P is the disjoint union of A, U, L, and N. Further, as P and Q are composed of the same elements and the sets A and A^* are equal, Q is the disjoint union of A^*, U, L, and N.

To create a bijection between the row-orbits of P and Q, we will first create a bijection between the order ideals of the subposets A and A^*. Our construction will rely on the following lemma:

Lemma 2.1.1. *Let P be a poset and P^d its dual. Let the bijection $c\colon J(P) \to J(P^d)$ be the complementation map that sends $I \mapsto P \backslash I$. Then the complementation map satisfies $c(\mathrm{row}_P(I)) = {\mathrm{row}_{P^d}}^{-1}(c(I))$ for all $I \in J(P)$.*

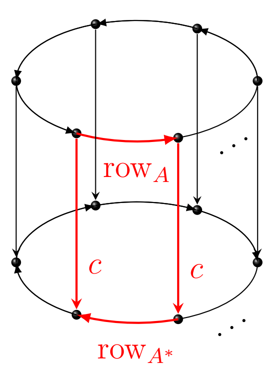
$$\begin{tikzcd}
I \arrow[r, mapsto, "\mathrm{row}_P"] \arrow[d, mapsto, "c"'] & I' \arrow[d, mapsto, "c"] \\
c(I) & c(I') \arrow[l, mapsto, "\mathrm{row}_{P^d}"]
\end{tikzcd}$$

Proof. Recall $\mathrm{row}_P = t_{v_1} \circ t_{v_2} \circ \cdots \circ t_{v_n}$ for some linear extension of P, $v_1 \leq v_2 \leq \cdots \leq v_n$. Reversing the order within the linear extension for P produces a linear extension for P^d, giving us $\mathrm{row}_{P^d} = t_{v_n} \circ \cdots \circ t_{v_2} \circ t_{v_1}$. This reversed ordering guarantees any element which changed status with respect to I due to its toggle in row_P will again change status via the toggle in row_{P^d} when performing $\mathrm{row}_{P^d}(c(\mathrm{row}_P(I)))$. Similarly, any element which is unchanged by row_P is unchanged by row_{P^d}. □

Note $A^* = A^d$, so by Lemma 2.1.1 the following diagram holds for all $I \in J(A)$ where $c\colon J(A) \to J(A^*)$ is the complementation map defined by $c(I) := A \backslash I$.

$$\begin{tikzcd}
I \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"'] & I' \arrow[d, mapsto, "c"] \\
c(I) & c(I') \arrow[l, mapsto, "\mathrm{row}_{A^*}"]
\end{tikzcd}$$

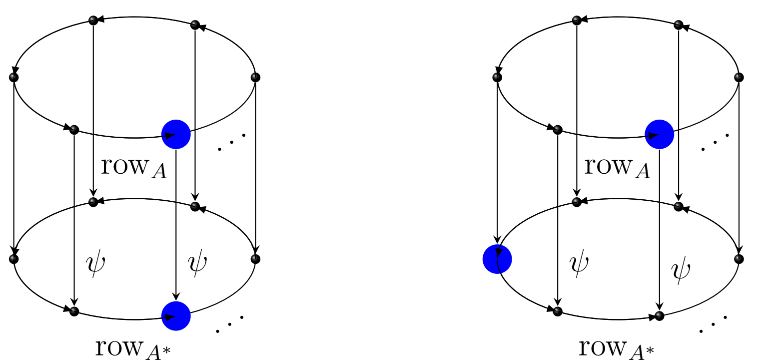
We now extend our view of these maps to the full row-orbits in A and A^*.



Above we have two cycles moving in opposite directions which are connected via the complementation bijection: rowmotion in A moving counter-clockwise and rowmotion in A^* moving clockwise. We will construct a new bijection $\psi\colon J(A) \to J(A^*)$ by flipping the row_{A^*}-orbit so that rowmotion moves counter-clockwise and then reconnecting the cycles, taking extra consideration with the row-orbit which contains the order ideals $I = \emptyset$ and A^*. In this case we note that $\mathrm{row}_A(A) = \emptyset$ and $\mathrm{row}_{A^*}(A^*) = \emptyset$. After flipping the row_{A^*}-orbit, we reconnect the cycles such that $\psi(\emptyset) = \emptyset$ and $\psi(A) = (A^*)$ as seen below.

$$\begin{tikzcd}
{} \arrow[r, mapsto, "\cdots"] & A \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"'] & \emptyset \arrow[r, mapsto, "\cdots"] \arrow[d, mapsto, "c"] & {} \\
{} & \emptyset \arrow[l, mapsto, "\cdots"'] & A^* \arrow[l, mapsto, "\mathrm{row}_{A^*}"] & {} \arrow[l, mapsto, "\cdots"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, mapsto, "\cdots"] & A \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "\psi"'] & \emptyset \arrow[r, mapsto, "\cdots"] \arrow[d, mapsto, "\psi"] & {} \\
{} \arrow[r, mapsto, "\cdots"] & A^* \arrow[r, mapsto, "\mathrm{row}_{A^*}"'] & \emptyset \arrow[r, mapsto, "\cdots"] & {}
\end{tikzcd}$$

In the case of the other row-orbits, the row_{A^*}-orbit, after being flipped, may be rotated any amount before reconnecting to the row_A-orbit. So both of the following choices for ψ are allowed, where the blue circles indicate an order ideal I and its complement $c(I)$ in A^*.



In both of these cases we have $\psi(\mathrm{row}_A(I)) = \mathrm{row}_{A^*}(\psi(I))$ for all $I \in J(A)$.

We will now extend ψ to the full posets, $\tilde{\psi}\colon J(P) \to J(Q)$. To this end, for each $I \in J(P)$, note that the elements of $I \backslash A$ form a subposet of P and there is an isomorphic subposet in Q made of the same elements. We will refer to both of these subposets by the name $I \backslash A$. We can now define the extended bijection to be $\tilde{\psi}(I) := (I \backslash A) \cup \psi(I \cap A)$ for all $I \in J(P)$.

Lemma 2.1.2. *Let $\tilde{\psi}\colon J(P) \to J(Q)$ be defined by $\tilde{\psi}(I) := (I \backslash A) \cup \psi(I \cap A)$ for all $I \in J(P)$. Then $\tilde{\psi}$ is a well-defined bijection.*

Proof. We begin with proving $\tilde{\psi}$ is well-defined, i.e., $\tilde{\psi}(I) \in J(Q)$ for all $I \in J(P)$. Let $I \in J(P)$. Let q be in the disjoint union $\tilde{\psi}(I) = (I \backslash A) \cup \psi(I \cap A) \subseteq Q$ and let $q' \in Q$ such that $q' \leq_Q q$. Then $\tilde{\psi}(I) \in J(Q)$ if and only if $q' \in \tilde{\psi}(I)$. We will proceed by cases.

Case 1*:* Let $q \in I \backslash A$ and let $q' \in A^*$. Then q must be in $U \subseteq Q$ and $U \subseteq P$. Furthermore, since A is autonomous, $I \cap A = A$. Then, given $\psi(A) = A^*$, $\tilde{\psi}(I) = (I \backslash A) \cup \psi(A) = (I \backslash A) \cup A^*$. Hence $q' \in \tilde{\psi}(I)$.

Case 2*:* Let $q \in I \backslash A$ and let $q' \notin A^*$. Recall the subposets $Q \backslash A^*$ and $P \backslash A$ are isomorphic, so $q' \leq_Q q \Rightarrow q' \leq_P q$. Thus $q' \in I \backslash A$. Consequently $q' \in \tilde{\psi}(I)$.

Case 3*:* Let $q \in \psi(I \cap A) \subseteq A^*$ and let $q' \in A^*$. Then $q' \leq_Q q \Rightarrow q' \leq_{A^*} q$ and $\psi(I \cap A) \in J(A^*)$, so $q' \in \psi(I \cap A) \subseteq \tilde{\psi}(I)$.

Case 4*:* Let $q \in \psi(I \cap A) \subseteq A^*$ and let $q' \notin A^*$. Then $q' \in L$ and $\tilde{\psi}(I) \cap A^* \neq \emptyset$. Since $\psi(\emptyset) = \emptyset$, $I \cap A \neq \emptyset$. Therefore, as A is autonomous, $I \supseteq L$. Hence $q' \in I \backslash A \subseteq \tilde{\psi}(I)$.

We will now show $\tilde{\psi}$ is a bijection. Let $\Psi \colon J(Q) \to J(P)$ be the mapping defined by $I' \mapsto (I' \backslash A^*) \cup \psi^{-1}(I' \cap A^*)$. Then

$$\Psi \circ \psi(I) = \Psi((I \backslash A) \cup \psi(I \cap A)) = (I \backslash A) \cup (I \cap A) = I,$$

so $\Psi \circ \psi = \mathrm{id}_{J(P)}$. Furthermore,

$$\psi \circ \Psi(I') = \psi((I' \backslash A^*) \cup \psi^{-1}(I' \cap A^*)) = (I' \backslash A^*) \cup (I' \cap A^*) = I',$$

so $\psi \circ \Psi = \mathrm{id}_{J(Q)}$. Therefore $\Psi = \tilde{\psi}^{-1}$ and $\tilde{\psi}$ is a bijection. □

Now that we have a bijection between the order ideals of P and Q, we are ready to discuss the main result of this section, Proposition 2.1.3.

Proposition 2.1.3 ([11], Proposition 4.10)**.** *Let P and Q be posets with* $\mathrm{com}(P) \simeq \mathrm{com}(Q)$*. Then there is a bijection φ between the row-orbits of $J(P)$ and the row-orbits of $J(Q)$ such that for any row-orbit $\mathcal{O} \subseteq J(P)$ we have:*
(1) $\#\mathcal{O} = \#\varphi(\mathcal{O})$;
(2) $\sum_{I \in \mathcal{O}} \mathrm{ddeg}(I) = \sum_{I \in \varphi(\mathcal{O})} \mathrm{ddeg}(I)$.

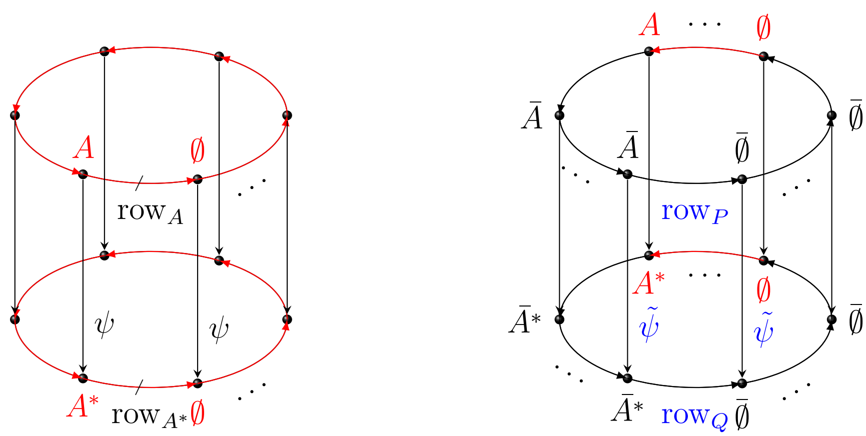
Proof. Let P and Q be posets with $\mathrm{com}(P) \simeq \mathrm{com}(Q)$. Then there is a bijection $\tilde{\psi} \colon J(P) \to J(Q)$ defined by $\tilde{\psi}(I) := (I \backslash A) \cup \psi(I \cap A)$ for all $I \in J(P)$, following our construction above. We will use $\tilde{\psi}$ to define the desired bijection φ.

To that end, we first need to better understand the construction of $\tilde{\psi}$. So, we will examine the intersection of an order ideal $I \in J(P)$ with A across a row-orbit $\mathcal{O} \subseteq J(P)$. As we move along $\mathcal{O}$ the intersection looks like the following sequence:

$$\ldots, I', \mathrm{row}_A(I'), {\mathrm{row}_A}^2(I'), \ldots, {\mathrm{row}_A}^{-1}(I'), \bar{A}, \bar{A}, \ldots, \bar{\emptyset}, \bar{\emptyset}, \ldots, \bar{\emptyset}, \ldots$$

where I' is the intersection $I \cap A$ with the lowest cardinality for all $I \in \mathcal{O}$, and the intersection is overlined if either $I \cap U \neq \emptyset$ or $I \cap L \neq L$. Note there might not be any overlined intersections $I \cap A$, or conversely we could have a constant overlined pattern $\bar{A}, \bar{A}, \ldots$ or $\bar{\emptyset}, \bar{\emptyset}, \ldots$ (see Example 2.1.4); nevertheless the not-overlined $I \cap A$'s always decompose into full row_A-orbits since the minimal elements of $(P \backslash I) \cap A$ are the same as the minimal elements of $A \backslash (I \cap A)$.

In the case of a constant overlined pattern across $\mathcal{O} \subseteq J(P)$, the bijection $\tilde{\psi}$ sends the order ideal $I \in \mathcal{O}$ to the order ideal composed of the same elements $I' \in J(Q)$. The anti-chain of maximal elements of these order ideals, which determine the behavior of rowmotion, always belong to the subposet $I \backslash A$ which isomorphic across P and Q, so clearly $\tilde{\psi}(\mathrm{row}_P(I) = \mathrm{row}_Q(\tilde{\psi}(I))$ for all $I \in \mathcal{O}$ where $\mathcal{O}$ has a constant intersection with A. Now consider the cases where the intersection $I \cap A$ passes through a row_A-orbit. There are two possibilities here: either there are overlined intersections or there are not. If there are overlined intersections, the row_A-orbit which is passed through is that which contains $I = \emptyset, A$. In this case, the row_{A/A^*}-orbits are split at the point sending $A(^*)$ to $\emptyset$ and the portion of the row orbit with an overlined intersection is inserted. This is depicted below with the node labels on the right corresponding to the intersections $I \cap A$, not the full order ideal.



For the orbits which have no overlined intersections, the row_P-orbit's intersections pass through the row_A-orbit a number of times (this could be once or multiple times depending on the elements of N, see Example 2.1.5), and the analogous statement holds in Q. The structure of the row_A and row_{A^*} orbits and their linkage via ψ is cleary preserved in this case, so $\tilde{\psi}(\mathrm{row}_P(I) = \mathrm{row}_Q(\tilde{\psi}(I))$ for all $I \in J(P)$. Thus by setting $\varphi(\mathcal{O}) := \{\tilde{\psi}(I) : I \in \mathcal{O}\}$ for any row_P-orbit $\mathcal{O} \subseteq J(P)$, we get a bijection of rowmotion orbits satisfying (1).

To show that (2) is also satisfied we claim something stronger holds: that for row_P-orbits, $\mathcal{O} \subseteq J(P)$, $\sum_{I \in \mathcal{O}} \mathcal{T}_{p^-}(I) = \sum_{I \in \varphi(\mathcal{O})} \mathcal{T}_{p^-}(I)$ for any $p \in P$. For any $p \notin A$, clearly $\mathcal{T}_{p^-}(I) = \mathcal{T}_{p^-}(\tilde{\psi}(I))$ for all $I \in J(P)$, immediately giving us what we want. So suppose $p \in A$. We will once again utilize the sequence of the intersection $I \cap A$ across $\mathcal{O}$. For any overlined $I \cap A$, $\mathcal{T}_{p^-}(I) = \mathcal{T}_{p^-}(\tilde{\psi}(I)) = 0$, so these may be ignored. Now consider a not-overlined row_A-orbit

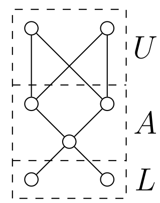
$$\mathcal{O}' = \{I', \mathrm{row}_A(I'), {\mathrm{row}_A}^2(I'), \ldots, {\mathrm{row}_A}^{-1}(I')\}.$$

Due to our construction of ψ, we have $\{\psi(I') : I' \in \mathcal{O}'\} = \{c(I') : I' \in \mathcal{O}'\}$. Note that p can be toggled out of I' if and only if p can be toggled into $c(I') \in J(A^*)$. Furthermore, for $I \in \mathcal{O}$ where $I \cap A$ is not overlined, p can be toggled out of I if and only if p can be toggled out of $I \cap A \in J(A)$; and similarly for toggling p into

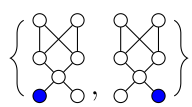
$\tilde{\psi}(I) \in J(Q)$ and $\tilde{\psi}(I) \cap A^* \in J(A^*)$. So for the orbits with not-overlined row_A-orbit intersections we have $\sum_{I\in\mathcal{O}} \mathcal{T}_{p^-}(I) = \sum_{I\in\varphi(\mathcal{O})} \mathcal{T}_{p^+}(I)$. By lemma 2.0.13, we know $\mathcal{T}_p$ averages to zero along any row-orbit, so $\sum_{I\in\mathcal{O}} \mathcal{T}_{p^-}(I) = \sum_{I\in\varphi(\mathcal{O})} \mathcal{T}_{p^-}(I)$ for all $p \in P$. □

Example 2.1.4. In this example, we will examine row-orbits in which all the order ideals have equal intersections with an autonomous subset. Consider the poset P below with the autonomous subset A and the subsequent sets U and L designated by the dashed lines.

P =

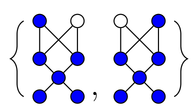


The following is a row-orbit which always has an empty intersection with A and thus a constant $\bar{\emptyset}$ pattern:



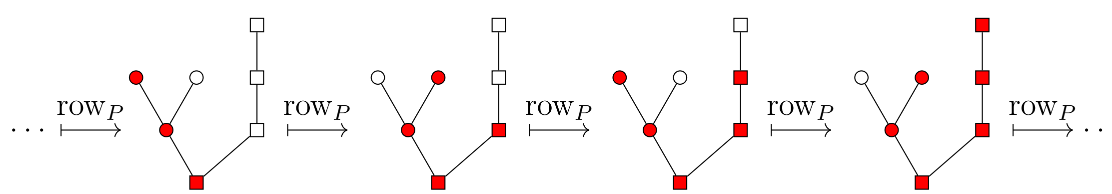
$$= \left\{ \bar{\emptyset}, \bar{\emptyset} \right\}.$$

This next row-orbit is an example of having a constant intersection $I \cap A = A$, giving us a constant $\bar{A}$ pattern:

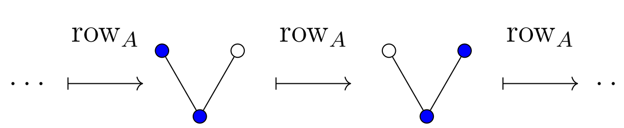


$$= \left\{ \bar{A}, \bar{A} \right\}.$$

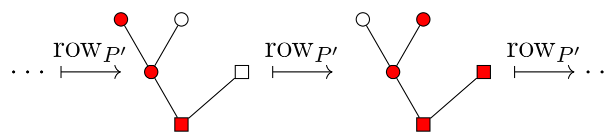
Example 2.1.5. In this example, we explore how incomparable elements affect the number of times a row-orbit's intersection passes through the row-orbit of an autonomous subposet. Consider the following row-orbit $\mathcal{O} \subseteq J(P)$. The elements of an autonomous subset $A \subseteq P$ are represented by circles and order ideals are shown in red.



The row_A-orbit through which the intersection $I \cap A$ moves is shown below.

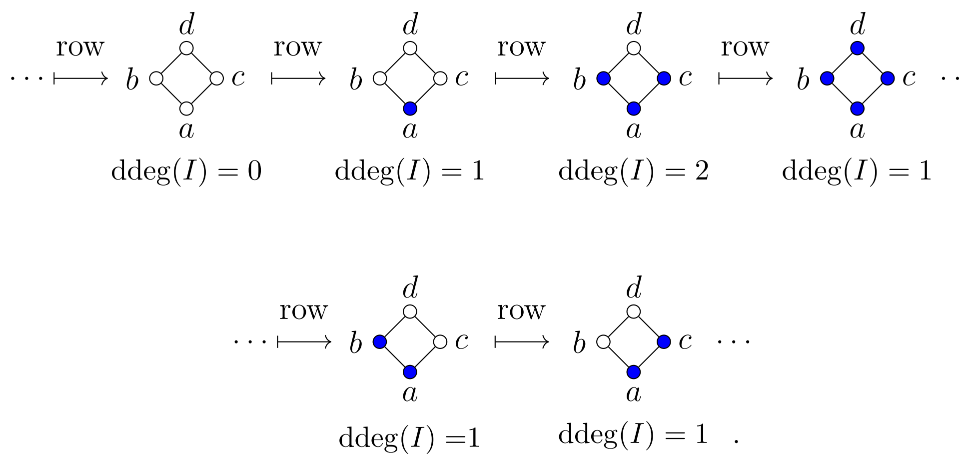


In this case we see the row_A-orbit occurs twice as we move through $\mathcal{O}$ due to the progression through the elements incomparable with A. Now let's change the number of incomparable elements and see what happens.

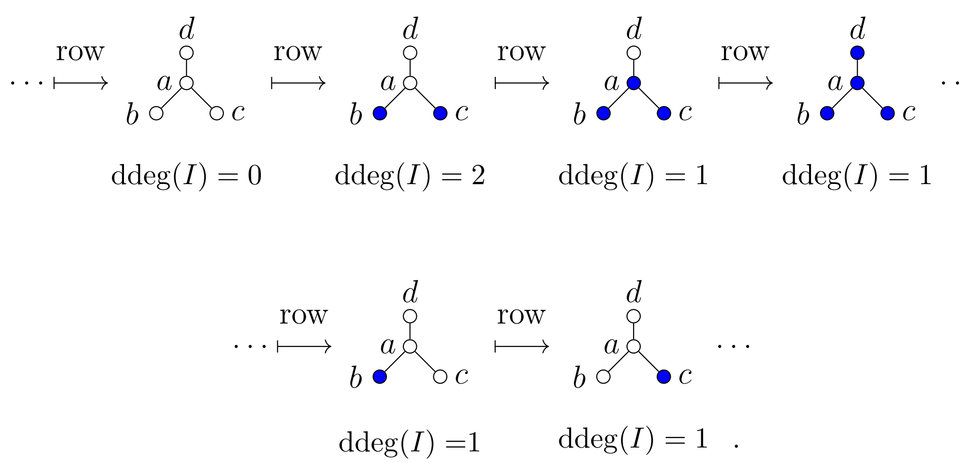


Here we see the row_A-orbit only occurs once as we move through the row-orbit $\mathcal{O}'$.

Example 2.1.6. In examining property (2) of Proposition 2.1.3, a natural question emerges: Why find a bijection such that the down-degree sums across the row-orbits are equal instead of each order ideal along the orbits having equal down-degree? To answer this question, consider the row-orbits of the following poset P:



Now we look at the row-orbits of the poset Q obtained from P by dualizing the autonomous subset $A = \{a, b, c\}$:



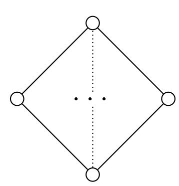
At first glance it may appear that we could have a bijection which sends order ideals of equal down-degree to each other given that the orbits are constructed of the same number of $\mathrm{ddeg}(I) = 0, 1, 2$. However, looking at the orbits of length 4, we note that for P the order ideal with $\mathrm{ddeg}(I) = 0$ is next to order ideals with $\mathrm{ddeg}(I) = 1$ on either side while in Q, there is a $\mathrm{ddeg}(I) = 2$ next to the $\mathrm{ddeg}(I) = 0$. This means there is no bijection that preserves the down-degree of individual order ideals that would also preserve the row-orbit structure.

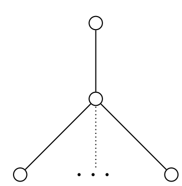
2.2 Rowmotion on P-partitions

In this section we will be broadening Proposition 2.1.3 from the context of order ideals to that of P-partitions. Recall that P-partitions of height $\ell = 1$ are the same thing as order ideals. Thus Proposition 2.1.3 is the $\ell = 1$ case of the following conjecture.

Conjecture 2.2.1 ([11], Conjecture 4.38)**.** *Let P and Q be posets with $\mathrm{com}(P) \simeq \mathrm{com}(Q)$. Let $\ell \geq 1$. Then there is a bijection φ between the row-orbits of $PP^\ell(P)$ and the row-orbits of $PP^\ell(Q)$ such that for any row-orbit $\mathcal{O} \subseteq PP^\ell(P)$ we have:*
(1) $\#\mathcal{O} = \#\varphi(\mathcal{O})$;
(2) $\sum_{T\in\mathcal{O}} \mathrm{ddeg}(T) = \sum_{T\in\varphi(\mathcal{O})} \mathrm{ddeg}(T)$.

We will be proving Conjecture 2.2.1 for the doppelgänger infinite families pictured below which we will refer to as diamond and broom posets. These posets are of size $n \geq 4$, where the broom poset (on the right) is obtained from the diamond poset (on the left) by dualizing the autonomous subset composed of the bottom and middle row of the diamond poset. By Lemma 2.0.7, for a diamond poset P and a broom poset Q of equal size n, $\mathrm{com}(P) \simeq \mathrm{com}(Q)$, so by Theorem 2.0.4 P and Q are doppelgängers.





Remark 2.2.2 ([11], Remark 4.19)**.** Recall that for $I \in J(P)$, $\mathrm{row}(I)$ is the unique order ideal of P which satisfies $\max(\mathrm{row}(I)) = \min(P\backslash I)$. This can be generalized for PL rowmotion acting on P-partitions. In particular, for $T \in PP^\ell(P)$, $\mathrm{row}^{PL}(T)$ is the unique element of $PP^\ell(P)$ with

$$\biguplus_{i=0}^{\ell-1} \max(\mathrm{row}^{PL}(T^{-1}(\{0,1,\ldots,i\}))) = \biguplus_{i=0}^{\ell-1} \min(P\backslash T^{-1}(\{0,1,\ldots,i\})), \tag{2.1}$$

where $\biguplus$ denotes multiset sum. Notice the left-hand side is equal to the down-degree of $\text{row}^{PL}(T)$ since

$$\begin{aligned}
\text{ddeg}(T) &= \sum_{i=0}^{\ell-1} \text{ddeg}(T^{-1}\{0,1,\dots,i\}) && \text{(by definition)} \\
&= \sum_{i=0}^{\ell-1} \#\max((T^{-1}\{0,1,\dots,i\}) && \text{(alternative definition for ddeg}(I)) \\
&= \#\biguplus_{i=0}^{\ell-1} \max(T^{-1}(\{0,1,\dots,i\})).
\end{aligned}$$

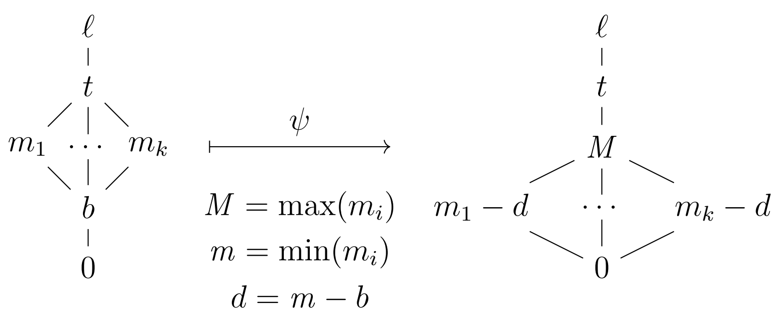
On the right-hand side, we find that an element $p \in P$ is in $\min(P \backslash T^{-1}(\{0,1,\dots,i\}))$ as many times as all elements below p are within the order ideal $T^{-1}(\{0,1,\dots,i\})$ while p is not. In other words, $p \in \min(P \backslash T^{-1}(\{0,1,\dots,i\}))$ for i in the range $\max(T(\downarrow p)) \leq i \leq T(p)$. This means that p will contribute $T(p) - \max(T(\downarrow p))$ to the multiset sum. We can then rewrite Equation 2.1 as

$$\text{ddeg}(\text{row}^{PL}(T)) = \sum_{p \in P} [T(p) - \max(T(\downarrow p))].$$

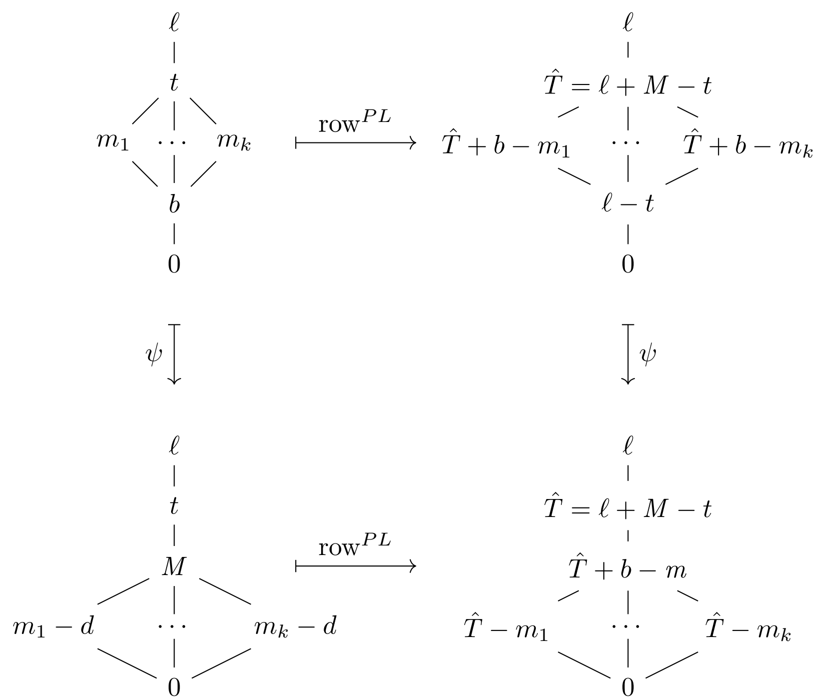
Proposition 2.2.3. *Let P be a diamond poset and Q be a broom poset with* $\text{com}(P) \simeq \text{com}(Q)$*. Let $\ell \geq 1$. Then there is a bijection φ between the row-orbits of $PP^{\ell}(P)$ and the row-orbits of $PP^{\ell}(Q)$ such that for any row-orbit $\mathcal{O} \subseteq PP^{\ell}(P)$ we have:*
(1) $\#\mathcal{O} = \#\varphi(\mathcal{O})$;
(2) $\sum_{T \in \mathcal{O}} \text{ddeg}(T) = \sum_{T \in \varphi(\mathcal{O})} \text{ddeg}(T)$.

Proof. Let P be a diamond poset and Q be a broom poset, both with size n. As stated above, since Q is obtained from P by dualizing the autonomous subset A composed of the bottom and middle rows of P, by Lemma 2.0.7, $\text{com}(P) \simeq \text{com}(Q)$.

We begin by defining the bijectiion $\psi \colon PP^{\ell}(P) \to PP^{\ell}(Q)$ as depicted below on the extended posets $\hat{P}$ and $\hat{Q}$.



Upon considering piecewise-linear rowmotion on the P-partitions as shown below, we find ψ satisfies $\psi(\text{row}^{PL}(T)) = \text{row}^{PL}(\psi(T))$ for all $T \in PP^{\ell}(P)$.



Then by setting $\varphi(\mathcal{O}) := \{\psi(T) : T \in \mathcal{O}\}$ for any row_P-orbit $\mathcal{O} \subseteq PP^\ell(P)$, we get a bijection of row-orbits satisfying (1).

Now let's show (2) is also satisfied. By Remark 2.2.2 we know $\mathrm{ddeg}(\mathrm{row}^{PL}(T)) = \sum_{p \in P}[T(p) - \max(T(\downarrow p))]$. Accordingly, for $T \in PP^\ell(P)$,

$$\begin{aligned} \mathrm{ddeg}(\mathrm{row}^{PL}(T)) &= t - M + \sum_{i=1}^{k} m_i - \sum_{i=1}^{k-1} b \\ &= t - M + \sum_{i=1}^{k} m_i - (k-1)b. \end{aligned} \tag{2.2}$$

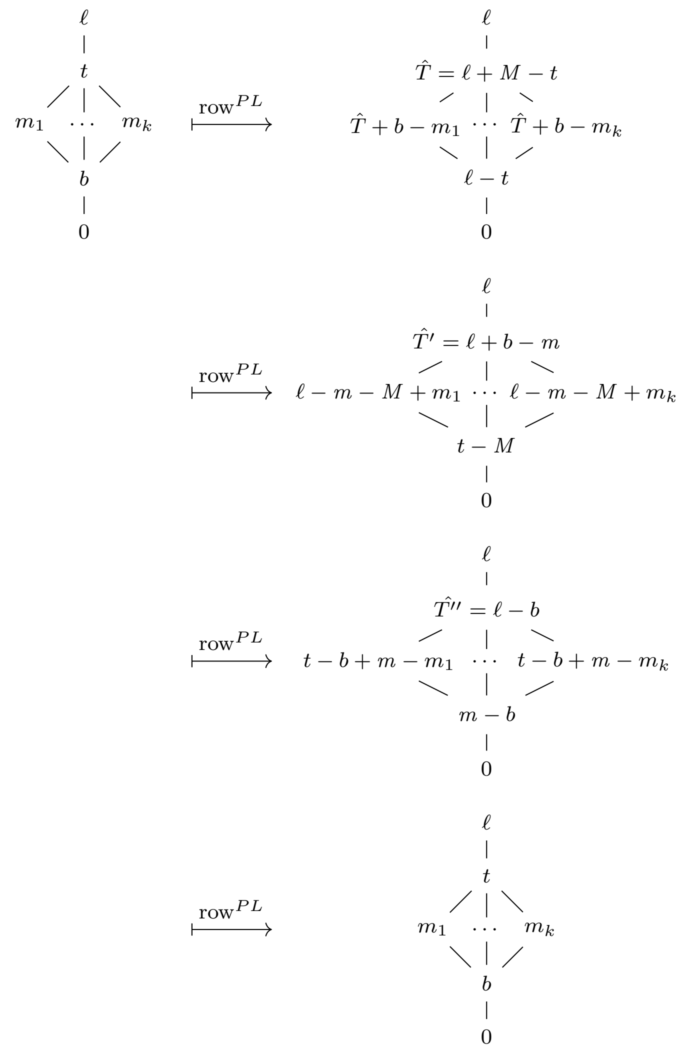
Furthermore, for $\psi(T) \in PP^\ell(Q)$,

$$\begin{aligned} \mathrm{ddeg}(\mathrm{row}^{PL}(\psi(T))) &= t - M + \sum_{i=1}^{k}(m_i - d) \\ &= t - M + \sum_{i=1}^{k} m_i - (k-1)m + (k-1)b. \end{aligned} \tag{2.3}$$

We then find the difference between the down-degrees in Equations 2.2 and 2.3 to be

$$\begin{aligned} \mathrm{ddeg}(\mathrm{row}^{PL}(\psi(T))) - \mathrm{ddeg}(\mathrm{row}^{PL}(T)) &= [t - M + \sum_{i=1}^{k} m_i - (k-1)m + (k-1)b] \\ &\qquad - [t - M + \sum_{i=1}^{k} m_i - (k-1)b] \\ &= -(k-1)m + 2(k-1)b \\ &= (k-1)(2b - m). \end{aligned}$$

For (2) to be satisfied, the difference $(k-1)(2b-m)$ must sum to zero across any row_P-orbit. In particular we claim $\sum_{T\in\mathcal{O}}(2b_T - m_T) = 0$ for any $\mathcal{O} \subseteq PP^{\ell}(P)$. Below is an arbitrary row_P-orbit $\mathcal{O} \subseteq PP^{\ell}(P)$ which shows PL rowmotion acting on the diamond poset has order 4.

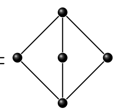
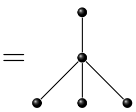


Consider the sum $\sum_{T\in\mathcal{O}}(2b_T - m_T)$ for the above orbit.

$$\begin{aligned}\textstyle\sum_{T\in\mathcal{O}}(2b_T - m_T) &= (2b-m) + (2(\ell - t) - (\hat{T} + b - M)) + (2(t-M) - (\ell - M)) \\ &\quad + (2(m-b) - (t - b + m - M)) \\ &= (2b-m) + (\ell - t - b)) + (2t - M - \ell) + (m - b - t + Mt) \\ &= 0.\end{aligned}$$

Clearly the difference in down-degrees between T and $\psi(T)$ sums to zero around an orbit of length 4. Given PL rowmotion acting on the diamond poset has order 4, we also know orbits may only be of lengths 1, 2, or 4. Since performing PL rowmotion as shown above on orbits of length 1 and 2 would simply be passing through the orbit multiple times, we can conclude that $\sum_{T\in\mathcal{O}}(2b_T - m_T) = 0$ for any $\mathcal{O} \subseteq PP^{\ell}(P)$, thus (2) is also satisfied. $\square$

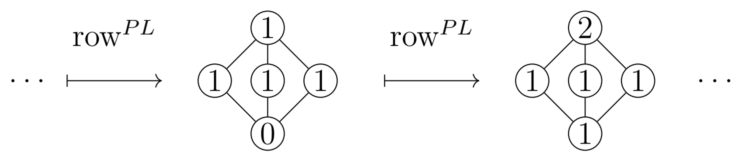
Example 2.2.4. In Example 2.1.6, we explored why we are looking for bijections between row-orbits with equal sums of down-degrees for order ideals. In that example we showed row-orbits which had the same frequency of down-degrees, but that need not be the case. Let's look at the diamond and broom posets of size $n = 5$ with P-partitions of height $\ell = 2$.

$P =$  $Q =$ 

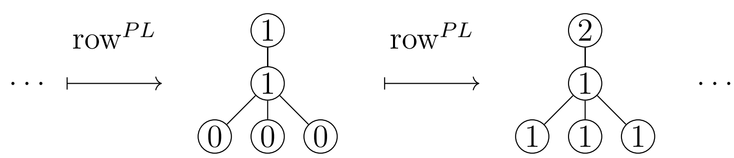
For both of these posets, there are:

- six row-orbits of length 4 with a summed down-degree of 11,
- two orbits of length 4 with a summed down-degree of 10,
- six orbits of length 2 with a summed down-degree of 6,
- one orbit of length two with a summed down-degree of 5.

The orbits of length 2 and summed down-degree of 5 clearly need to map to each other for the bijection to fulfill the desired properties. Looking at these row-orbits we see row-orbits in bijection are not necessarily composed of the same down-degrees in varying order, but can have P-partitions with entirely different down-degrees.



ddeg$(T) = 2$ ddeg$(T) = 3$



ddeg$(T) = 4$ ddeg$(T) = 1$

The bijection we found between the row-orbits of the broom and diamond poset families is constructed in a manner not generalizable to all posets with isomorphic comparability graphs. Furthermore, our proof of bijected orbits having equal down-degree sums relied on direct calculation across an arbitrary orbit of maximum length, which is also not generalizable and would be cumbersome for posets where rowmotion has large order. One would hope there is a more natural method for finding a bijection with the desired properties. In the next chapter, we explore moving between polytopes associated to our posets in the hopes of finding a more widely applicable method for constructing a suitable bijection.

Chapter 3

Moving to the Chain Polytope

In addition to the order polytope, there are other convex polytopes associated with our poset P. Of particular interest for the study of doppelgänger pairs is the *chain polytope*, which we will be studying in an effort to find an alternative proof method for Proposition 2.2.3 that could apply to the more general Conjecture 2.2.1.

Definition 3.0.1. The **chain polytope** $\mathscr{C}(P)$ of a poset P is the subset of $\mathbb{R}^P$ defined by the following conditions:

$$\begin{aligned} 0 \leq g(x) & \qquad \text{for all } x \in P \\ g(y_1) + \cdots + g(y_k) \leq 1 & \qquad \text{for every } \textit{maximal} \text{ chain } y_1 < \cdots < y_k \text{ of } P. \end{aligned}$$

Analogous to Theorem 1.2.2 for the order polytope, there is a simple description for the vertices of $\mathscr{C}(P)$.

Theorem 3.0.2 ([19], Thm. 2.2)**.** *The vertices of $\mathscr{C}(P)$ are the characteristic functions χ_A of antichains A of P. In particular, the number of vertices of $\mathscr{C}(P)$ is the number of antichains of P.*

Turning briefly to graph theory, we note that antichains A of P have one-to-one correspondence to independent sets in the comparability graph $\mathrm{com}(P)$, making $\mathscr{C}(P)$ the so-called vertex packing polytope of $\mathrm{com}(P)$. The vertex packing polytope of a graph being the convex hull of its independent sets. Thus $\mathscr{C}(P)$ only depends on the comparability graph of P. Consequently, posets with isomorphic comparability graphs also have isomorphic chain polytopes.

The structures of $\mathcal{O}(P)$ and $\mathscr{C}(P)$ are related. There is a bijection sending a filter S to the antichain $A = \min\{S\}$, so from Theorem 1.2.2 and Theorem 3.0.2 we find $\mathcal{O}(P)$ and $\mathscr{C}(P)$ have the same number of vertices. In general, however, $\mathcal{O}(P)$ and $\mathscr{C}(P)$ do not have to be combinatorially equivalent. The theorem below describes one class of posets for which the order polytope and chain polytope are combinatorially equivalent.

Theorem 3.0.3 ([19], Thm. 2.3)**.** *Suppose P has no three-element chains (i.e., P has length at most one). Then $\mathcal{O}(P)$ and $\mathscr{C}(P)$ are affinely equivalent and hence combinatorially equivalent.*

Stanley defined a *transfer map* $\phi\colon \mathcal{O}(P) \to \mathscr{C}(P)$ that allows us to transfer certain properties of the order polytope to the chain polytope.

Definition 3.0.4 ([19], Def. 3.1 and Thm. 3.2)**.** Let P be a finite poset. Then the **transfer map** $\phi\colon \mathcal{O}(P) \to \mathscr{C}(P)$ is defined as follows: If $f \in \mathcal{O}(P)$ and $x \in P$, then

$$(\phi f)(x) = \min\{f(x) - f(y) : x \text{ covers } y \text{ in } P\}$$

with the inverse given by

$$(\phi^{-1} g)(x) = \max\{g(y_1) + g(y_2) + \cdots + g(y_k) : \hat{0} \lessdot y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k = x\}.$$

Theorem 3.0.5 ([19], Thm. 3.2)**.**

(a) The transfer map $\phi\colon \mathcal{O}(P) \to \mathscr{C}(P)$ is a continuous, piecewise-linear bijection.

(b) Let ℓ be a positive integer and $f \in \mathcal{O}(P)$. Then $\ell f(x) \in \mathbb{Z}$ for all $x \in P$ if and only if $\ell(\phi f)(x) \in \mathbb{Z}$ for all $x \in P$.

From the second part of the above theorem, we find a relation between the Ehrhart polynomials of $\mathcal{O}(P)$ and $\mathscr{C}(P)$ that is fundamental to the classification of doppelgängers.

Theorem 3.0.6 ([19], Thm. 4.1)**.** *The Ehrhart polynomials of $\mathcal{O}(P)$ and $\mathscr{C}(P)$ are given by*

$$i(\mathcal{O}(P), \ell) = i(\mathscr{C}(P), \ell) = \Omega_P(\ell + 1).$$

We utilized this fact throughout Chapter 2 when finding doppelgänger pairs via their isomorphic comparability graphs (Thm. 2.0.4), as the chain polytope depends only on $\mathrm{com}(P)$ and therefore, through the above theorem, the order polynomial $\Omega_P(\ell)$ depends solely on $\mathrm{com}(P)$. (Recall doppelgänger pairs are defined by having equal order polynomials). This theorem also tells us that the number of linear extensions of P, the number of $PP^{\ell}(P)$, and the volumes of $\mathcal{O}(P)$ and $\mathscr{C}(P)$ depend only on the comparability graph of P as well.

In order to define rowmotion on the chain polytope, we first need to define a third polytope associated with our poset P: the *order-reversing polytope*.

Definition 3.0.7 ([12], Def. 3.3)**.** The **order-reversing polytope** $\mathcal{OR}(P)$ of a poset P is the subset of $\mathbb{R}^P$ defined by the following inequalities:

$$\begin{aligned} 0 \le f(x) \le 1 &\qquad \text{for all } x \in P \\ f(x) \ge f(y) &\qquad \text{for } x \le y \text{ in } P. \end{aligned}$$

Elements $f \in \mathcal{OR}(P)$ are extended to the poset $\hat{P}$ with the convention that $f(\hat{0}) = 1$ and $f(\hat{1}) = 0$.

Theorem 3.0.8 ([12])**.** *The vertices of $\mathcal{OR}(P)$ are the characteristic functions χ_I of order ideals I of P. In particular, the number of vertices of $\mathcal{OR}(P)$ is the number of order ideals of P.*

Just as there is a bijection between filters and antichains, there is a bijection from order ideals to antichains: $I \mapsto \max(I)$. Therefore the polytopes $\mathcal{O}(P)$, $\mathscr{C}(P)$, and $\mathcal{OR}(P)$ all have the same number of vertices.

We will define rowmotion on the three associated polytopes as the composition of three maps as done by Joseph in [12]. One we have already defined—the transfer map; we will now define the two others.

Definition 3.0.9. The **complementation map** $c\colon \mathcal{O}(P) \to \mathcal{OR}(P)$ is defined by $f(x) \mapsto 1 - f(x)$. (We also use c to denote the inverse function from $\mathcal{OR}(P)$ to $\mathcal{O}(P)$.)

Definition 3.0.10 ([12], Prop. 3.6)**.** The bijection $\mathrm{OR}\colon \mathscr{C}(P) \to \mathcal{OR}(P)$ is given by

$$(\mathrm{OR}(g))(x) = \max\{g(y_1) + g(y_2) + \cdots + g(y_k) : x = y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k \lessdot \hat{1}\}$$

with inverse given by

$$(\mathrm{OR}^{-1}(f)(x) = \min\{f(x) - f(y) : y \in \hat{P}, y \gtrdot x\} = f(x) - \max\{f(y) : y \in \hat{P}, y \gtrdot x\}.$$

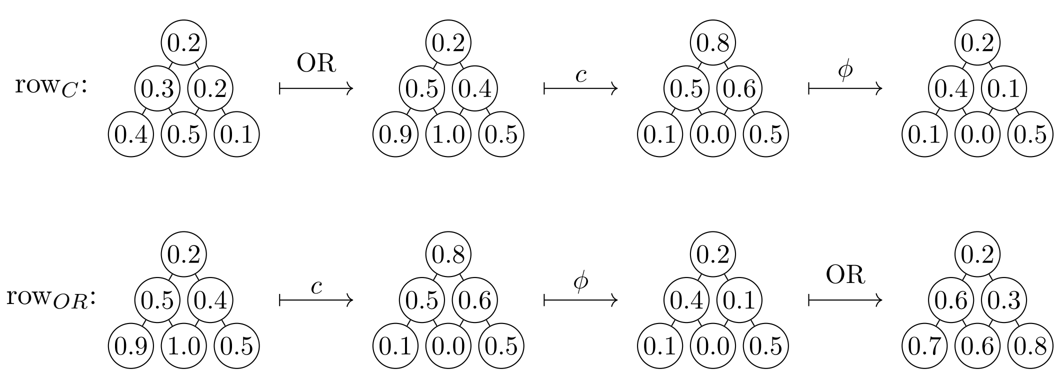
Notice OR is just the inverse of the transfer map applied to the dual poset.

Finally, we are able to define our rowmotion operators.

Definition 3.0.11 ([12], Def. 3.8)**.** Let row_C, row_{OR}, row_{OP} be defined as follows:

$$\begin{aligned}
\mathrm{row}_C\colon \mathscr{C}(P) &\xrightarrow{\mathrm{OR}} \mathcal{OR}(P) \xrightarrow{c} \mathcal{O}(P) \xrightarrow{\phi} \mathscr{C}(P)\\
\mathrm{row}_{OR}\colon \mathcal{OR}(P) &\xrightarrow{c} \mathcal{O}(P) \xrightarrow{\phi} \mathscr{C}(P) \xrightarrow{\mathrm{OR}} \mathcal{OR}(P)\\
\mathrm{row}_{OP}\colon \mathcal{O}(P) &\xrightarrow{c} \mathcal{OR}(P) \xrightarrow{\mathrm{OR}^{-1}} \mathscr{C}(P) \xrightarrow{\phi^{-1}} \mathcal{O}(P).
\end{aligned}$$

Example 3.0.12. Below we demonstrate row_C and row_{OR}.



The mapping row_{OP} is equivalent to $\mathrm{row}^{PL^{-1}}$, which is the notation we will continue to use. From the above definitions, we find that $\mathrm{row}^{PL^{-1}}$ is equal to row_{OR} for the dual poset and the commutative diagrams in Figure 3.1 hold.

There is another way to move from PL row-orbits on the order polytope to row-orbits on the chain polytope that has an added benefit in the ease of calculating the down-degree of P-partitions. This mapping is Hopkins' modified version of Stanley's transfer map.

$$
\begin{tikzcd}
\mathcal{O}(P) \arrow[d, "\mathrm{dual}"'] & \mathcal{O}(P) \arrow[l, "\mathrm{row}^{PL}"'] \arrow[d, "\mathrm{dual}"] \\
\mathcal{OR}(P^d) \arrow[r, "\mathrm{row}_{OR}"] \arrow[d, "\mathrm{OR}^{-1}"'] & \mathcal{OR}(P^d) \arrow[d, "\mathrm{OR}^{-1}"] \\
\mathscr{C}(P^d) \arrow[r, "\mathrm{row}_C"'] & \mathscr{C}(P^d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(P) \arrow[r, "\mathrm{row}^{PL}"] \arrow[d, "c"'] \arrow[ddr, "\phi"] & \mathcal{O}(P) \arrow[d, "c"] \\
\mathcal{OR}(P) \arrow[d, "\mathrm{OR}^{-1}"'] & \mathcal{OR}(P) \arrow[d, "\mathrm{OR}^{-1}"] \\
\mathscr{C}(P) \arrow[r, "\mathrm{row}_C"'] & \mathscr{C}(P)
\end{tikzcd}
$$

Figure 3.1: Commutative diagrams of mappings between the order, order-reversing, and chain polytopes for a poset P and its dual.

Definition 3.0.13 ([11], Section 2)**.** Hopkins defines a transfer map $\tilde{\phi}\colon \mathcal{O}(P) \to \mathscr{C}(P)$ by

$$(\tilde{\phi} f)(x) = \begin{cases} 1 - f(x) & \text{if } x \text{ is maximal in } P; \\ \min\{f(y) : y \in P \text{ covers } x\} - f(x) & \text{otherwise.} \end{cases}$$

The above changed transfer map definition is the result of a composition—actually there are two compositions which produce this map. As Hopkins described the difference as "essentially given by replacing P by P^d" ([11], footnote 4), we will start by describing that path of mappings. One of the functions it utilizes is a natural isomorphism across the chain polytopes of doppelgänger pairs with isomorphic comparability graphs: $g \in \mathscr{C}(P) \mapsto g' \in \mathscr{C}(Q)$ where $g'(x) := g(x)$ for all x in P and Q. In other words, we consider the posets as being constructed of the same elements (as done in Chapter 2) and maintain the element labels. Recall that a poset P and its dual P^d always have isomorphic comparability graphs. We can now show the full composition:

$$\tilde{\phi}\colon \mathcal{O}(P) \xrightarrow{\mathrm{dual}} \mathcal{OR}(P^d) \xrightarrow{c} \mathcal{O}(P^d) \xrightarrow{\phi} \mathscr{C}(P^d) \xrightarrow{\sim} \mathscr{C}(P).$$

So for $f \in \mathcal{O}(P)$ and $x \in P$ we have

$$\begin{aligned} f(x) \xmapsto{\mathrm{dual}} f(x) \xmapsto{c} 1 - f(x) \xmapsto{\phi} & (1 - f(x)) - \max\{(1 - f(y)) : y \lessdot x \text{ in } P^d\} \\ &= \begin{cases} 1 - f(x) & \text{if } x \in \min(P^d); \\ \min\{f(y) : y \gtrdot x \text{ in } P\} - f(x) & \text{else} \end{cases} \\ &\xmapsto{\sim} \begin{cases} 1 - f(x) & \text{if } x \in \max(P); \\ \min\{f(y) : y \gtrdot x \text{ in } P\} - f(x) & \text{else.} \end{cases} \end{aligned}$$

The second composition which equals $\tilde{\phi}$ is simpler:

$$\tilde{\phi}\colon \mathcal{O}(P) \xrightarrow{c} \mathcal{OR}(P) \xrightarrow{\mathrm{OR}^{-1}} \mathscr{C}(P).$$

For $f \in \mathcal{O}(P)$ and $x \in P$ we have

$$f(x) \overset{c}{\mapsto} 1 - f(x) \xmapsto{\mathrm{OR}^{-1}} \begin{cases} 1 - f(x) & \text{if } x \in \max(P); \\ \min\{f(y) : y \gtrdot x \text{ in } P\} - f(x) & \text{else.} \end{cases}$$

This path appears in Figure 3.1 where we see $\tilde{\phi}$ sends P-partitions to the same row-orbit as ϕ. Regardless of the method of achieving the modified definition, the resulting version of the transfer map maintains the properties listed in Theorem 3.0.5 which made the original definition so useful. Now we come to the added benefit in calculating P-partition down-degrees.

Lemma 3.0.14. *Let $T \in PP^\ell(P)$. Then*

$$\frac{1}{\ell}\,\mathrm{ddeg}(T) = \sum_{p \in P} \tilde{\phi}(\frac{1}{\ell}T)(p).$$

It is an easy check to see that the map $\tilde{\phi}$ sends each scaled label $\frac{1}{\ell}T(x)$ to the number of times x is counted as a maximal element (scaled by $\frac{1}{\ell}$) within the order ideals $T^{-1}\{0, 1, \ldots, i\}$ for $i = 0, \ldots, \ell - 1$. Since $\tilde{\phi}$ and ϕ map to the same row-orbit, this method of calculating down-degree will also be useful when considering the sum of down-degrees across row-orbits utilizing Stanley's transfer map.

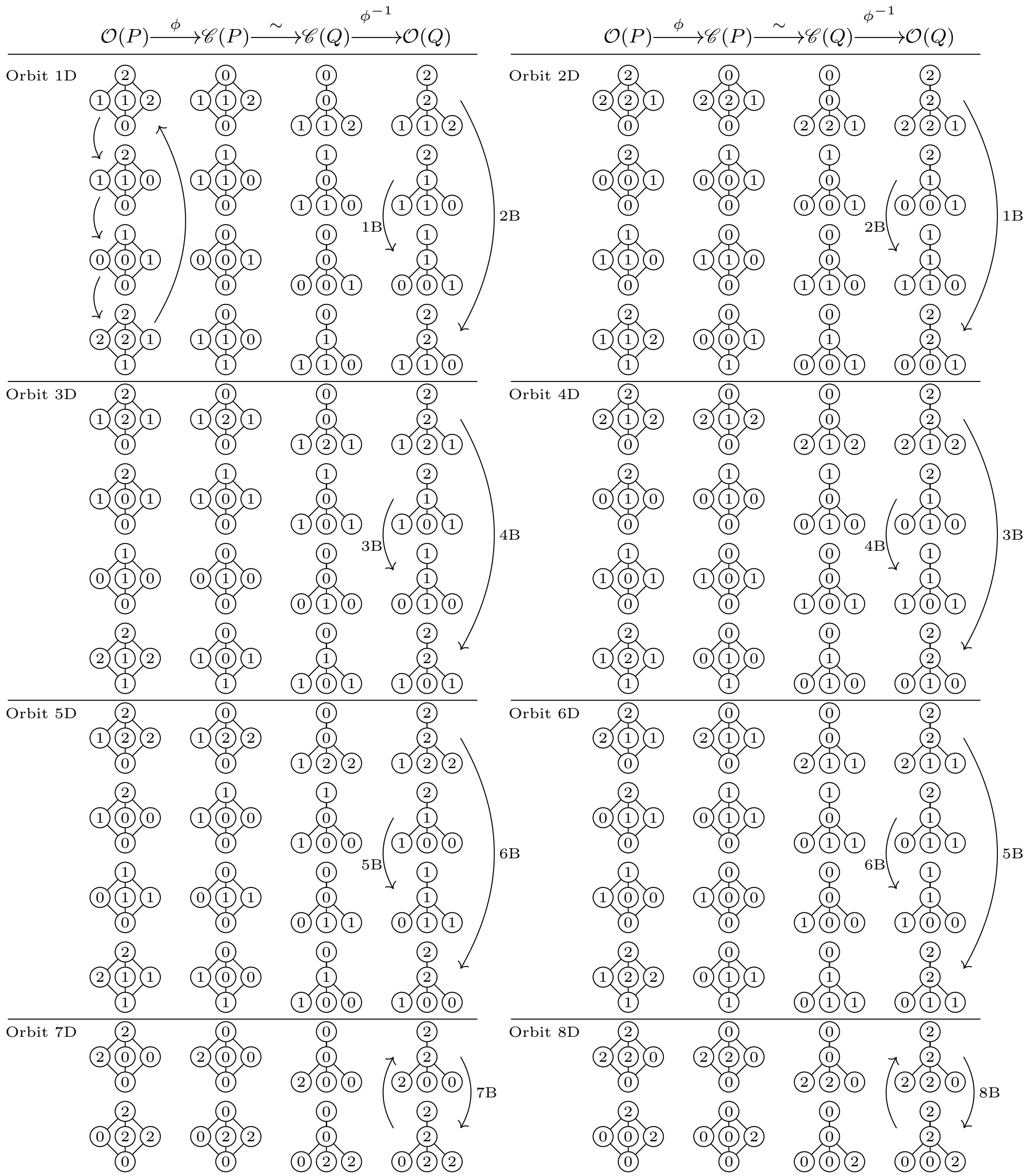
We now have defined multiple methods for mapping between the order polytope and chain polytope for a given poset, each of which preserves the row-orbit structure. In Example 3.0.15, we look at a bijection to and across the chain polytopes for a pair of posets with isomorphic comparability graphs in the hopes of finding a way to send orbits to orbits. To do so, we utilized Stanley's transfer map and the natural isomorphism between chain polytopes previously described.

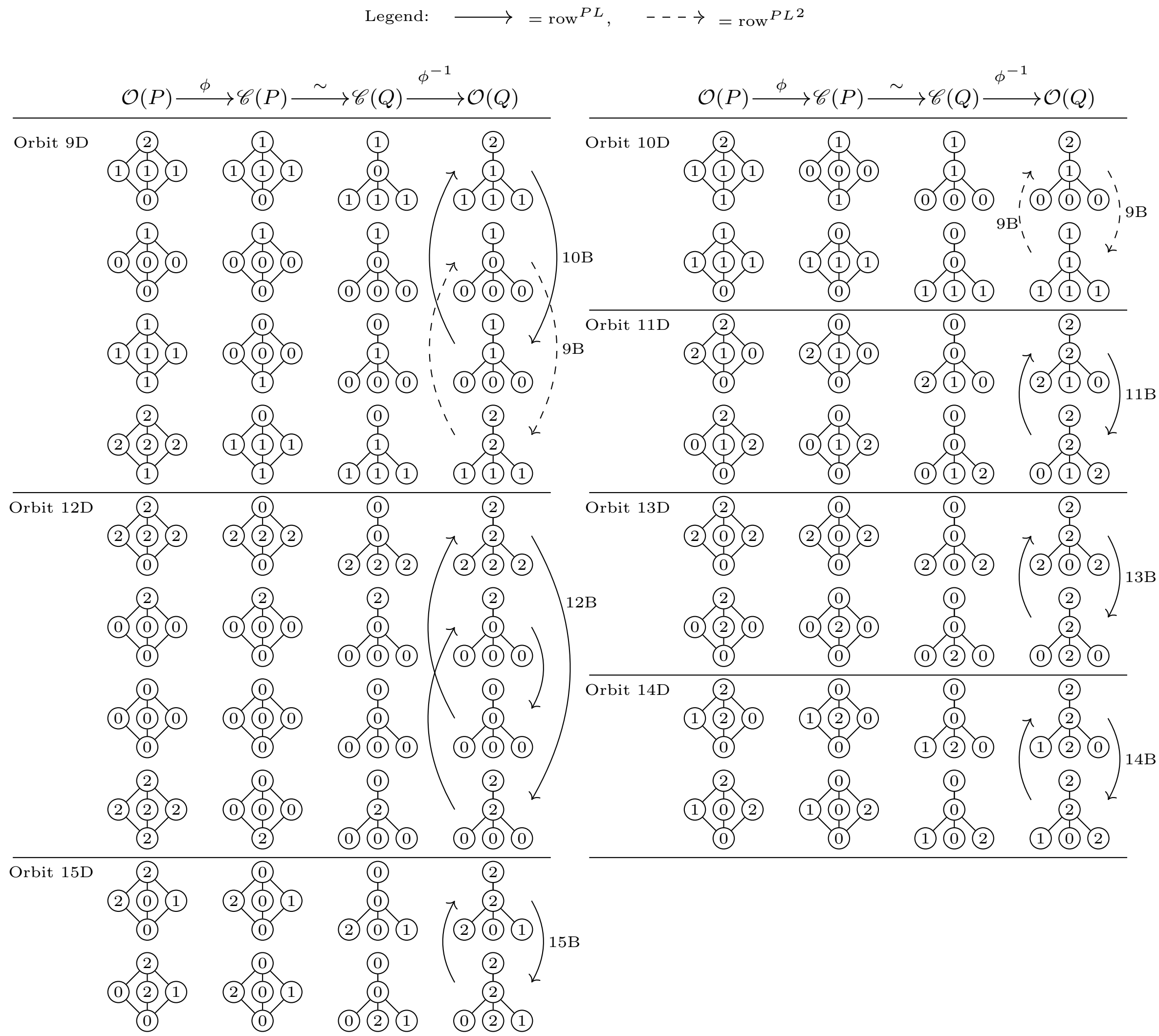
Example 3.0.15. Let P be the diamond poset of size 5, and let Q be the broom poset of size 5. Then P and Q are doppelgängers with isomorphic comparability graphs and, therefore, isomorphic chain polytopes. The following table shows P-partitions $T \in PP^2(P)$ being mapped to P-partitions $T' \in PP^2(Q)$. The mapping moves to and across the chain polytopes via Stanley's transfer map and the natural isomorphism described above. The table consists of two large columns, each with four subcolumns showing the results of an intermediary bijection between polytopes. The left-most subcolumns, labelled $\mathcal{O}(P)$, contain the 15 row-orbits for P. Each of these orbits is labelled by a number and the letter D as in diamond.

Orbit 1D, the first row_P-orbit in the table, has arrows indicating the path through which the P-partitions cycle under PL rowmotion: top-to-bottom. All other row_P-orbits are laid out in the same manner. In the right-most subcolumn corresponding to orbit 1D, there are P-partitions from two distinct row_Q-orbits. The arrows labelled 1B and 2B (B as in broom) designate rowmotion to and from P-partitions within the 1B and 2B row_Q-orbits respectively. The row_P and row_Q-orbits with the same number in their labels are those paired by the bijection constructed in Proposition 2.2.3; for example $\varphi(1D) = 1B$.

Moving to the second large column, we find orbit 2D maps to the other halves of the 1B and 2B orbits. The orbits 1B and 2B cycle through their respective P-partitions in the first column top-to bottom and then the second column top-to-bottom. This pattern holds for the other orbits which were similarly split and paired: 3B, 4B, 5B, and 6B. In the case of the orbit 9B, which split such that the first and second columns each contain a row^{PL^2}-orbit, the P-partitions cycle from the top in the first column, to the bottom in the second, to the bottom in the first, and the top of the second. All other row_Q-orbits are fully outlined by arrows.

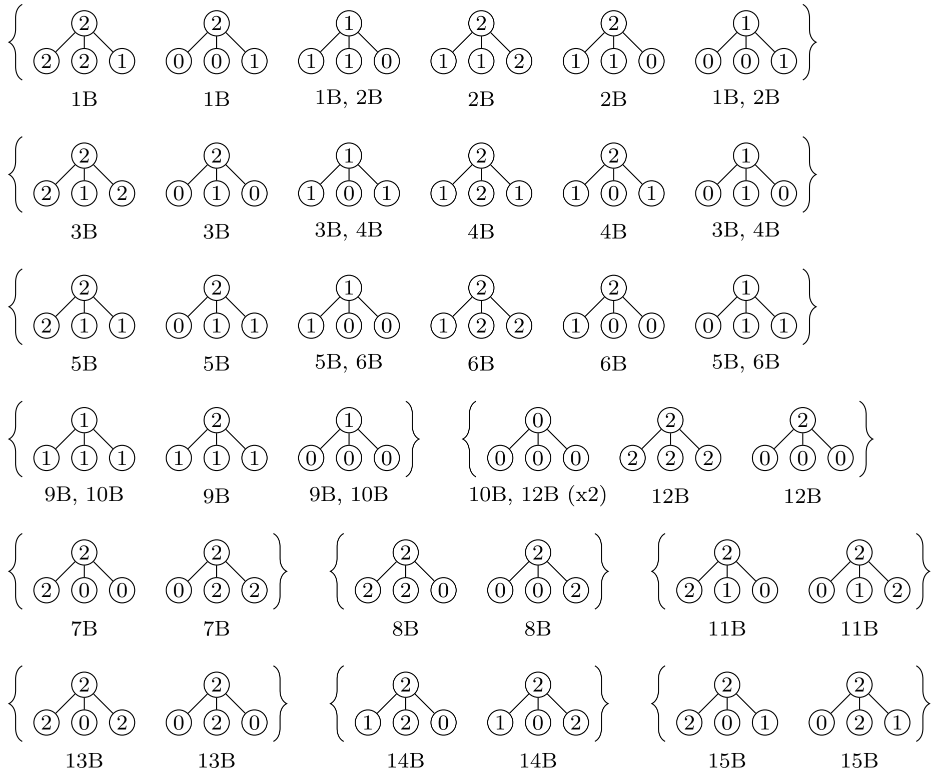
Legend: $\longrightarrow = \text{row}^{PL}$, $\dashrightarrow = \text{row}^{PL^2}$





In the above table, we find row-orbits of P are either being mapped to their corresponding orbits found via the bijection in Proposition 2.2.3 (although not necessarily preserving the row-orbit structure, see orbit 12D) or they are paired with another orbit on P and sent to two P-partitions from each of the corresponding orbits on Q. (For example, 1D and 2D are each sent to two P-partitions from 1B and 2B). These pairings of P's row-orbits are determined by the row-orbits of the autonomous subset A.

Listed below are the 11 row_{A^*}-orbits. Underneath each P-partition T are the orbits which contain a P-partition with T as a subset.



In the row_{A^*}-orbits of length six, we see pairings that appeared under the bijection across the polytopes: 1B/2B, 3B/4B, and 5B/6B. The P-partitions which show up in two row_Q-orbits have multiple options for the value of the element in $Q\backslash A^*$ (if a P-partition labels the top element of A^* one, then the element in $Q\backslash A^*$ could be labelled one or two). The special case of 9B/10B also appears in the row_{A^*}-orbits. When looking at the P-partitions which appear in the orbit 10B we see three which also appear in another orbit. Furthermore, we come across the unique case of a P-partition which appears as a subset three times, the all-zero partition. Here we see the pairing under the bijection across the polytopes appears when the row_{A^*}-orbit has multiple P-partitions which are shared; 9B and 10B are paired in the bijection, while 10B and 12B are not. We also note that where the all-zero partition in 12B repeats is where the row-orbit structure was broken by our bijection across the chain polytopes. Finally, observe that all the row_{A^*}-orbits of length two contain P-partitions which appear as subsets in only one row_Q-orbit. These are the row-orbits which were paired in the same manner by the above bijection and the bijection from Proposition 2.2.3.

The transfer map is known to preserve row-orbit structure (see Figure 3.1), so in Example 3.0.15, the bijection which disrupts some of the orbit structures is the natural isomorphism used to cross the chain polytopes. All of the other mappings we have defined to move from the order polytope to the chain polytope—ϕ, $\tilde{\phi}$, and $\mathrm{OR}^{-1} \circ$ dual—also preserve orbit structure. When used in conjunction with the natural isomorphism to move across the polytopes, the result has the same paired orbits as found above (the specific P-partitions are not always bijected in the same way, rather orbit 1D still maps to P-partitions from orbits 1B and 2B, etc.). In other words, the natural isomorphsim across the chain polytopes is not a component of a bijection that solves Conjecture 2.2.1 as we hoped. However, the highly structured manner in which it manipulates row_Q-orbits based on the row-orbits of A leads us to believe that future work could construct a bijection between the isomorphic chain polytopes that preserves the row-orbit structure.

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