M-matrices and Chip-Firing

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David Perkinson

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$$\forall u, \quad 10v - \frac{2i}{3} > -2(u - 5v).$$

Table of Contents

Abstract

This thesis examines the relationship between the abelian sandpile model and M-matrices. Equivalent conditions for M-matrices are examined using chip firing techniques.

To the memory of Suzanne Shafer. ♡

Introduction

Originally introduced by Bak, Tang and Wiesenfeld [1] in the context of self-organized criticality and the abelian sandpile model by Dhar [5], chip-firing games are well known in the field of combinatorics. Similarly, originally introduced by Ostrowski [13], M-matrices have played pivotal roles in many different disciplines including Economics, Statistics, and Physics. In [9], Gabrielov showed the firing rules of the abelian sandpile model are modeled by M-matrices. Building upon Gabrielov's work, Guzmán and Klivans [10] further explored chip-firing on M-matrices, extending the concept of energy minimizing configurations to M-matrices.

Interest in M-matrices arose during the thesis process in an attempt to better understand the abelian sandpile model on a hexagonal lattice with triangular boundary – presented in the final chapter of this thesis. During that effort we built upon the theory developed by Gabrielov [9] and Guzmán and Klivans [10]. This thesis is primarily focused on reframing some of the conditions of M-matrices into the language of chip-firing games.

In the first section we define configurations, script firing, and the graph associated with a given Z-matrix. We walk through the process of stabilization for configurations on Z-matrices and show that they are not necessarily avalanche finite. In the second section we introduce M-matrices, a subclass of Z-matrices. We show that nonsingular real M-matrices can stabilize any configuration. The third section presents the algebraic structure of recurrent configurations on graphs described by M-matrices. In the second chapter we explore symmetric configurations on M-matrices and present a conjecture for the all-ones configuration on the hexagonal lattice with triangular boundary.

Before we begin, it is important to have some intuition regarding chip-firing on graphs. We will quickly introduce the abelian sandpile model. Let $G = (V, E)$ be a connected directed graph with finite vertex set V and edge multiset E. The weight function on edges is defined as

$$\mathrm{wt}(u, v) = \#\text{ of edges from } u \text{ to } v,$$

for $(u, v) \in E$. If $(u, v) \notin E$, then $\mathrm{wt}(u, v) = 0$. The *indegree* and *outdegree* of a vertex play an important role in our study of chip-firing games. They are found as follows:

$$\mathrm{indegree}(v) = \sum_{w} \mathrm{wt}(w, v),$$

$$\text{outdegree}(v) = \sum_{w} \text{wt}(v, w).$$

Let us consider a chip-firing game played on G. Suppose exactly one vertex has outdegree equal to 0 and assume all vertices have a directed path to this vertex. We refer to this vertex as the *sink*. To play the chip firing game, assume each non-sink vertex has a nonnegative number of chips resting on it. If a given vertex has more chips than its outdegree we say it is legal to fire that vertex. When a vertex v fires, it loses as many chips as its outdegree, while its neighbors gain a chip for each edge coming from v. By restricting the sink so that it never fires, we can guarantee that this system will eventually stabilize, meaning that no vertex can fire.

Example 0.1. Figure 1 shows the process of chip firing on the house graph. Note that the final configuration of chips has no vertices that can legally fire.

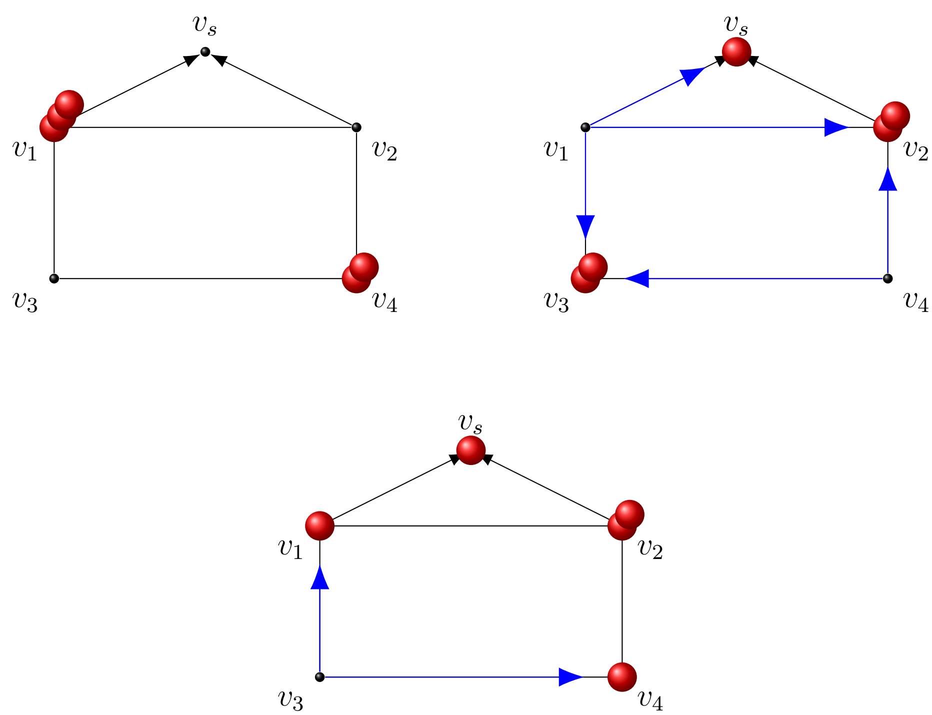


Figure 1: Chip-firing on the house graph with sink v_s.

Chapter 1

Chip-firing Systems

1.1 Z-Matrices

Definition 1.1. An $n \times n$ real matrix A is called an *Z-matrix* if A has nonpositive off-diagonals.

Example 1.2. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$. Then A is a Z-matrix if $b, c \leq 0$.

The directed graph $\Gamma(A)$ associated to a Z-matrix A is the graph with vertex set $[n] \cup \{s\}$ where s is disjoint from $[n]$ and is called the *sink vertex*. The graph $\Gamma(A)$ has directed edges: (a) (i, j) if $A_{ji} \neq 0$ and $i \neq j$ and (b) (j, s) if $\sum_i A_{ij} \neq 0$. The *weight* of an edge (i, j) in case (a) is defined to be $\mathrm{wt}(i, j) = -A_{ji} \geq 0$ and the weight of an edge (j, s) in case (b) is $\mathrm{wt}(j, s) = \sum_i A_{ij}$, which may be negative. For clarity, we will sometimes denote vertex i by v_i.

A *configuration* c on $\Gamma(A)$ is a vector in $\mathbb{R}^n$. We denote the i-th component of c by c_i. A configuration's *degree* is $\deg(c) = \sum_i c_i$. We think of c_i as the number of chips on vertex i and $\deg(c)$ as the total number of chips on $[n]$. Note that we allow configurations with both positive and negative amounts of chips that can be non-integer valued, though we will mostly be imagining configurations that have nonnegative integer components. We say a configuration is *unstable* at a vertex $v \in [n]$ if $c_v \geq A_{vv}$ and *stable* otherwise. If a configuration is unstable at any vertex, it is called *unstable*; otherwise it is called *stable*.

If $c \in \mathbb{R}^n$ and $v \in [n]$ we can *fire* or *topple* v to obtain a new configuration $\tilde{c} = c - Ae_v$, where e_v is the v-th standard basis vector. To denote the process of toppling we write $c \xrightarrow{v} \tilde{c}$. The toppling is *legal* if c_v is unstable. The process of consecutive legal topplings of vertices is called an *avalanche*.

Suppose $\tilde{c}$ and c are configurations and $\sigma \in \mathbb{Z}^n_{\geq 0}$ such that $\tilde{c} = c - A\sigma$. We call σ a *firing script* taking c to $\tilde{c}$ and write $c \xrightarrow{\sigma} \tilde{c}$. We say σ is a *legal firing script* if there exists a sequence $v_1, v_2, \ldots, v_k$ of legal vertex firings with $\sigma = \sum_i e_{v_i}$. If c is a configuration and there exists a legal firing script σ such that $c - A\sigma$ is stable, then we say c is *stabilizable* with *stabilization*

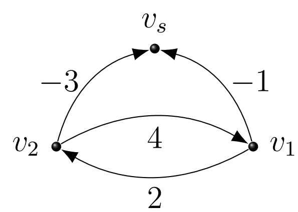
$$c^\circ := c - A\sigma.$$

Note that, for configurations c and d on some $\Gamma(A)$, if $d \geq c$ componentwise and d is stablizable then c is as well. The Z-matrix A is called *avalanche finite* if every configuration on $\Gamma(A)$ is stabilizable. It is important to note that not all Z-matrices are avalanche finite. The following examples illustrates this point.

Example 1.3. Let

$$A = \begin{pmatrix} 1 & -4 \\ -2 & 1 \end{pmatrix}.$$

Then $\Gamma(A)$ is



Consider the unstable configuration $c = (1, 0)$ on $\Gamma(A)$, firing the first vertex yields the unstable configuration (0,2) and firing the second vertex yields (4,1). In general firing v_1 or v_2 will only add more chips to v_2 and v_1 respectively, at a faster rate than either vertex can get rid of them. So $(1, 0)$ will never stabilize; similarly, $(0, 1)$ is not stabilizable. The configuration (a, b) is stabilizable if and only if $a \leq 0$ and $b \leq 0$.

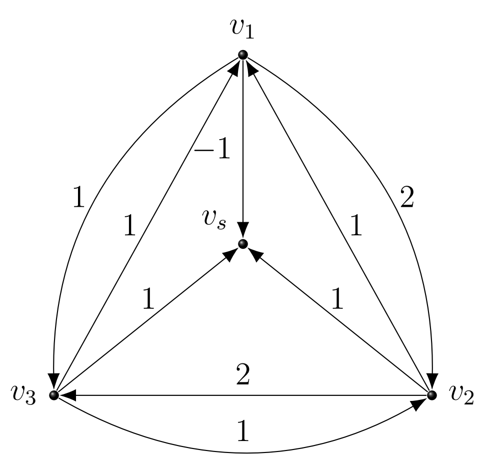


Figure 1.1: Directed graph $\Gamma(A)$ for Example 1.4.

Example 1.4. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

Then the directed graph represented by A appears in Figure 1.1. We will now step through the stabilization process of the configuration $c = (3,2,0)$ on $\Gamma(A)$.

$$c = (3,2,0) \xrightarrow{v_1} (1,4,1) \xrightarrow{v_2} (2,0,3) \xrightarrow{v_3} (3,1,0) \xrightarrow{v_1} (1,3,1) = c^\circ.$$

The corresponding firing script is $\sigma = (2,1,1)$ since we fired v_1 twice and vertices two and three once. So we have $c^\circ = c - A\sigma$.

Definition 1.5. The *Laplacian* L of a graph G is an $(n+1) \times (n+1)$ matrix that describes G. We number the vertices from 1 to $n+1$; it is convention to number the sink vertex $n+1$. Then we define the Laplacian to be

$$L_{ij} = \begin{cases} \text{outdegree}(v_i) & i = j, \\ -\text{wt}(v_j, v_i) & i \neq j. \end{cases}$$

The *reduced Laplacian* of the graph with respect to $n+1$ is obtained by deleting the row and column corresponding to $n+1$.

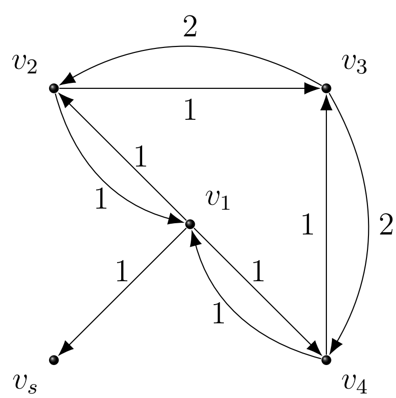


Figure 1.2: The graph G for Example 1.6.

Example 1.6. Consider the graph in Figure 1.2, the Laplacian of G is

$$L = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_s \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_s \end{array} \\ \begin{pmatrix} 3 & -1 & 0 & -1 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ -1 & 0 & -2 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

It is clear that L is a Z-matrix since the off-diagonals are negative or zero. To obtain the reduced Laplacian with respect to v_s we delete the row and column corresponding to v_s. Then

$$\tilde{L} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -2 & 2 \end{pmatrix} \end{array}.$$

This is a Z-matrix and we see that $\Gamma(\tilde{L}) = G$.

1.2 Avalanche Finiteness

Let $c, d \in \mathbb{R}^n$. Then if $c_i \geq 0$ for all $i \in [n]$ we say $c \geq 0$. Similarly we say $c > 0$ if $c_i > 0$ for all i. If we have $d_i \geq c_i$ or $d_i > c_i$ for all $i \in [n]$ we say $d \geq c$ or $d > c$, respectively. Now consider $A \in \mathbb{R}^{n \times n}$; we say $A \geq 0$ and call A a *nonnegative matrix* if for all $i, j \in [n]$, $A_{ij} \geq 0$.

Theorem 1.7 (Least Action Principle)**.** *Let c be an arbitrary configuration and let $\tau \geq 0$ be an integer vector such that $d = c - A\tau$ is stable. Then for any finite sequence of legal firings starting at c with script firing vector σ, we have $\tau \geq \sigma$.*

The least action principle is a powerful theorem that will be very useful in future proofs. It was originally proved by Diaconis and Fulton [6].

Proof. We will prove this by induction on the degree of the firing script, $\deg(\sigma) = \sum \sigma_i$. For $\deg(\sigma) = 0$ the statement is trivial. Assume that the result holds for all firing scripts of degree at most $k-1$ for some $k \geq 1$. Let $i_1, i_2, i_3, \ldots, i_k$ be a legal firing sequence with script σ. Since i_1 is unstable in c and $d = c - A\tau$ is stable we know that $\tau_{i_1} \geq 1$. Let $c' = c - Ae_{i_1}$, $\tau' = \tau - e_{i_1}$, and $\sigma' = \sigma - e_{i_1}$. Then $i_2, i_3, \ldots, i_k$ is a legal firing sequence for c' with script σ' and $d = c - A\tau = c' - A\tau'$ is stable. So by induction $\tau' \geq \sigma'$. Hence $\tau \geq \sigma$. □

Remark 1.8. Note that the least action principle implies that if $c - A\tau \geq 0$ is stable for any firing script $\tau \geq 0$, then c is stabilizable by a legal firing script σ, where $\tau \geq \sigma$.

Corollary 1.9. *If a configuration c is stabilizable then every avalanche starting at c is finite. Furthermore, every avalanche stabilizing c has the same firing script. Hence if c' and c'' are stable and can be reached from c by a legal firing script then $c' = c''$.*

Proof. This is immediate from the least action principle. □

We are now ready to introduce a specific subclass of Z-matrices that have a intriguing relationship to avalanche finitness.

Definition 1.10. Let A be an $n \times n$ matrix. Then the *spectral radius* of A is

$$\rho(A) = \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}.$$

Definition 1.11. An $n \times n$ matrix A is called an *M-matrix* if $A = s\mathbb{I} - B$ where B is a nonnegative matrix and s is some real number such that $s \geq \rho(B)$.

Much of the advancement of the theory of M-matrices was due to Perron's [14] and Frobenius' [8] seminal works regarding nonnegative matrices. The following theorem is a well-known result of those works.

Theorem 1.12 (Perron-Frobenius)**.** *Let B be a nonnegative matrix. Then there is a nonnegative real number λ_{pf}, called the* Perron root *or the* Perron-Frobenius eigenvalue*, such that $\lambda_{pf} \geq |\lambda|$ for any eigenvalue λ of B. Furthermore, there exists a eigenvector for λ_{pf} with nonnegative components.*

Theorem 1.13 (Jordan Form)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Then there exists an invertible matrix P such that $P^{-1}AP$ has the form*

$$\begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & J_k \end{pmatrix}.$$

Where each J_i is a Jordan Block *having the form*

$$\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix},$$

where λ is an eigenvalue of A.

Remark 1.14. Theorem 1.13 tells us that since any Z-matrix A can be represented as $A = s\mathbb{I} - B$ with $B \geq 0$, then for any A there exists a invertible matrix P such that $P^{-1}BP$ is in Jordan form. Then $P^{-1}AP = s\mathbb{I} - P^{-1}BP$, which is a block matrix with each block of the form

$$\begin{pmatrix} s-\lambda & -1 & 0 & \cdots & 0 \\ 0 & s-\lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & s-\lambda \end{pmatrix},$$

where λ is an eigenvalue of B.

Note that $P^{-1}AP$ is not the Jordan Form for A; however, $-P^{-1}AP$ is the Jordan Form for $-A$ with eigenvalues $\lambda - s$, where λ ranges over the eigenvalues of B. Then we can conclude that A has eigenvalues $s - \lambda$.

We get the following result.

Proposition 1.15. Let A be an M-matrix. Writing $A = s\mathbb{I} - B$ with $B \geq 0$ and $s \geq \rho(B)$, we have A is nonsingular if and only if $s > \rho(B)$.

Definition 1.16. An $n \times n$ matrix A is said to be *convergent* if $\lim_{m\to\infty} A^m = 0_{n\times n}$.

Theorem 1.17. *An $n \times n$ matrix A is convergent if and only if $\rho(A) < 1$.*

Proof. Without loss of generality we may assume A is in Jordan Form, in which case the result is straightforward. □

Corollary 1.18. *If $\rho(A) < 1$, then $\mathbb{I} - A$ is nonsingular with inverse $\sum_{m=0}^{\infty} A^m$.*

Proof. Let $\rho(A) < 1$, then A is convergent. For each $m \geq 0$ we have

$$(\mathbb{I} - A)(\mathbb{I} + A + \cdots + A^m) = (\mathbb{I} - A^{m+1}).$$

Taking the limit as $m \to \infty$ of $\mathbb{I} - A^{m+1}$ we obtain $\mathbb{I}$, so it follows that $(1 - A)^{-1}$ exists and is equal to $\sum_{m}^{\infty} A^m$. □

1.2.1 Structure of M-matrices

Plemmons [15] complied a list of 40 different characterizations of nonsingular M-matrices. Below we present proofs for the equivalence of four characterizations and in future sections will show the equivalence of two more. Of most importance from the point of view of this thesis, Gabrielov [9] has shown that a Z-matrix is avalanche finite if and only if it is a nonsingular M-matrix. That result appears as Theorem 1.22, below.

Theorem 1.19. *Let A be a Z-matrix. Then the following are equivalent:*

1. *A is a nonsingular M-matrix;*

2. *The real part of A's eigenvalues are positive;*

3. *A^{-1} exists and its entries are nonnegative.*

4. *$A\sigma \geq 0$ implies $\sigma \geq 0$, i.e., A is* monotone.

Remark 1.20. If A is a nonsingular M-matrix then A^t is an nonsingular M-matrix as well. A^t is a Z-matrix with spectrum equal to that of A.

Proof. $(1 \Leftrightarrow 2)$ $(\Rightarrow)$ Let A be an $n \times n$ nonsingular M-matrix and write $A = s\mathbb{I} - B$ with $B \geq 0$ and $s > \rho(B)$. From Remark 1.14 we know that A has eigenvalues of the form $s - \lambda$, where λ ranges over the eigenvalues of B. Since $s > \rho(B)$, $\mathrm{Re}(s - \lambda) > 0$ for all λ of B.
$(\Leftarrow)$ Suppose the eigenvalues of A have positive real parts and write $A = s\mathbb{I} - B$, where $B \geq 0$. Let λ_{pf} be the Perron root of B. Then λ_{pf} is real and $\lambda_{pf} \geq |\lambda| \geq \mathrm{Re}(\lambda)$ for every eigenvalue λ of B. Then since A has eigenvalues of the form $s - \lambda$ where $s \in \mathbb{R}$, with positive real part, we must have $s > \rho(B)$.

(1 ⇔ 3) (⇒) (As in Horn [12]) Let $A = s\mathbb{I} - B$, where $B \geq 0$ and $s > \rho(B)$. Consider $\frac{1}{s}A = \mathbb{I} - \frac{1}{s}B$. Since $s > \rho(B)$ we have that $0 \leq \rho\left(\frac{1}{s}B\right) < 1$. Then by Theorem 1.17, $\sum_{m=0}^{\infty} \frac{1}{s} B^m$ converges, which implies $\left(\mathbb{I} - \frac{1}{s}B\right)^{-1}$ exists. Since $B \geq 0$, we must have $\left(\frac{1}{s}A\right)^{-1} \geq 0$. Hence A^{-1} exists and is nonnegative.

(⇐) Suppose $A^{-1} \geq 0$, and write $A = s\mathbb{I} - B$ with $B \geq 0$. Let λ_{pf} be the Perron-Frobenious eigenvalue of B with corresponding eigenvector $\sigma \gneq 0$. Then

$$A\sigma = (s - \lambda_{pf})\sigma \Rightarrow \sigma = (s - \lambda_{pf})A^{-1}\sigma.$$

Since $A^{-1}\sigma \gneq 0$, it follows that $s > \lambda_{pf} = \rho(B)$. So A is a nonsingular M-matrix.

(3 ⇔ 4) (⇒) Assume $A\sigma = \tau \geq 0$ for some $\sigma \in \mathbb{R}^n$. Then $A^{-1}A\sigma = A^{-1}\tau$, and since τ and A^{-1} are both nonnegative, $A^{-1}\tau = \sigma \geq 0$.
(⇐) Let $A\sigma \geq 0$ imply $\sigma \geq 0$. We want to show that $\ker(A) = 0$, hence A^{-1} exists. Suppose $Au = 0$. Then $u \geq 0$. However, we also have $A(-u) = 0$; thus it must be the case that $-u \geq 0$. Therefore we must have that $u = 0$. To see that $A^{-1} \geq 0$ consider its j-th column $A^{-1}e_j$ for each $j \in [n]$. We have $A(A^{-1}e_j) = e_j \geq 0$, so by assumption $A^{-1}e_j \geq 0$. □

Remark 1.21. Let A be a nonsingular M-matrix and $A = B - C$ a splitting of A's diagonal and off-diagonal elements. Then $C \geq 0$. We have

$$\mathbb{I} = (B - C)A^{-1} = BA^{-1} - CA^{-1}.$$

Since $CA^{-1} \geq 0$ we must have $B > 0$. That is, all diagonal elements of a nonsingular M-matrix are positive.

Theorem 1.22. *A Z-matrix A is avalanche finite if and only if A is a nonsingular M-matrix.*

Proof. (⇒)(Due to Gabrielov [9]) Now assume A is an avalanche finite Z-matrix and $c \in \mathbb{R}^n_{\geq 0}$. For each $k = 1, 2, 3, \ldots$, let σ_k be the legal firing script stabilizing kc. Let $\tau_k = \sigma_k/k$. Then we have

$$k\sigma - A\sigma_k = (k\sigma)^\circ \Rightarrow A\tau_k = \sigma - \frac{(k\sigma)^\circ}{k}.$$

Each τ_k lies in a compact ball with respect to the L^1-norm on $\mathbb{R}^n$. To see this, let ω be the legal firing script stabilizing $c_{\max} + 1$. Necessarily we have $A\omega > 0$. Let $N = \max_i c_i$, then $c - NA\omega \leq 0$ and since k is positive $kc - kNA\omega \leq 0$, so by the least action principle $\sigma_k \leq N\omega$. This implies $0 \leq \tau_k \leq N\omega$, thus $|\tau_k| \leq |N\omega|$. So τ_k is bounded in a compact set as k varies. Now since τ_k is in a compact set there exists a subsequence τ_{k_i} such that $\tau_{k_i} \to \tau$ for some $\tau \in \mathbb{R}^n_{\geq 0}$. So

$$\begin{aligned} A\tau &= \lim_{i\to\infty} A\tau_{k_i} \\ &= \lim_{i\to\infty} \sigma - \frac{(k_i\sigma)^\circ}{k_i} \\ &= \sigma. \end{aligned}$$

Note that $\frac{(k\sigma)^\circ}{k}$ converges to 0 because $(k\sigma)^\circ$ stays bounded as k gets large. We have shown that for each $\sigma \geq 0$ there exists a $\tau \geq 0$ such that $A\tau = \sigma$. Applying this result with $\sigma = e_i$ for each standard basis vector e_i in turn shows that A^{-1} exists and is nonnegative.

($\Leftarrow$) Let A be a nonsingular M-matrix and c a configuration on $\Gamma(A)$. Then by Theorem 1.19, $A^{-1} \geq 0$. Let σ be a configuration such that $\sigma > c$ and $\sigma \geq 0$. Define $\tau = A^{-1}\sigma \geq 0$. Then $c - A\tau$ is stable. So by the least action principle, c is stabilizable. □

1.3 Recurrents and Superstables

This section extends the group structure of the abelian sandpile model, to our class of graphs $\Gamma(A)$ where A is a nonsingular matrix. Many of the proofs follow directly from Perkinson and Corry's forthcoming textbook [4].

1.3.1 Recurrent Configurations

Definition 1.23. A stable configuration c is *recurrent* if for every configuration a, there exists a configuration $b \geq 0$, such that $c = (a + b)^\circ$.

The only recurrent configuration that can always be found with ease is the maximal stable configuration, which we now define.

Definition 1.24. For an $n \times n$ Z-matrix A the *maximal stable configuration* on $\Gamma(A)$ is denoted $c_{\max}$ and is found by

$$(c_{\max})_v = A_{vv} - 1,$$

for all $v \in [n]$.

Theorem 1.25. *For any $\Gamma(A)$, $c_{\max}$ is recurrent.*

Proof. Let $a \geq 0$ be any configuration on $\Gamma(A)$, and $b = c_{\max} - a^\circ$. Say σ is a legal firing script that stabilizes a. Then σ is a legal firing script for $a + b$, thus $(a + b) \xrightarrow{\sigma} (a^\circ + b) = c_{\max}$. Since $c_{\max}$ is stable this means that $c_{\max} = (a + b)^\circ$, thus $c_{\max}$ is recurrent. □

In fact $c_{\max}$ can be beneficial at determining other recurrent configurations.

Theorem 1.26. *A configuration c is recurrent if and only if there exists a configuration $b \geq 0$, such that $c = (c_{\max} + b)^\circ$.*

Proof. ($\Rightarrow$) Let c be a recurrent configuration. Then by definition of recurrent, c is stable and there exists a configuration $b \geq 0$ such that $c = (c_{\max} + b)^\circ$.
($\Leftarrow$) Suppose there exists a $b \geq 0$ such that $c = (c_{\max} + b)^\circ$. Let a be any configuration such that σ_a stabilizes a and say $b' = c_{\max} - a^\circ + b \geq 0$. Then

$$a + b' \xrightarrow{\sigma_a} a^\circ + b' = c_{\max} + b.$$

So $(a + b')^\circ = c$. □

Example 1.27. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

Then $c_{\max} = (1,3,2)$. There are 7 other recurrent configurations on $\Gamma(A)$: $(1,2,2)$, $(1,1,2)$, $(1,0,2)$, $(0,2,2)$, $(1,3,0)$, $(0,3,2)$, $(0,3,1)$, $(1,3,1)$, $(0,3,0)$. The configuration $(0,3,1)$ is a recurrent since

$$c_{\max} + (0,0,1) = (1,3,3) \xrightarrow{v_3} (2,4,0) \xrightarrow{v_1} (0,6,1) \xrightarrow{v_2} \ldots \longrightarrow (0,3,1)$$

In the stabilization of $(1,3,3)$ we have that site 1 fires four times, site two fires 3 times and site 3 fires 4 times. The firing script that stabilizes $(1,3,3)$ is $\sigma = (4,3,4)$, so $(0,3,1) = (1,3,3) - A\sigma$.

Definition 1.28. The stable addition of configurations a and b is defined to be

$$a \circledast b = (a+b)^{\circ}.$$

One can quickly see that stable addition must be commutative with the all zero configuration being its identity element. Similarly, one can see that since a sequence of legal fires for configurations $a+b$ is legal for $a+b+c$ as well, stable addition must be associative. The set of stable configurations under stable addition does not form a group in general due to the lack of inverses. However, as shown below, restricting to the subset of recurrents a group is formed under stable addition.

Notation 1.29. We denote all linear integer combinations of the columns of a nonsingular M-matrix A by $\mathcal{A}$. That is

$$\mathcal{A} = \mathrm{Im}_{\mathbb{Z}}(A).$$

Theorem 1.30. *Let A be an $n \times n$ nonsingular M-matrix. Then the recurrent configurations $\mathcal{S}(A)$ form a group under stable addition, and*

$$\mathcal{S}(A) \to \mathbb{R}^n/\mathcal{A}$$

$$c \mapsto c + A$$

is an isomorphism of groups.

Remark 1.31. If we restrict A to the integers we recover the well-known mapping from the abelian sandpile model. That is

$$\mathcal{S}(A) \cap \mathbb{Z}^n \to \mathbb{Z}^n/\mathcal{A}.$$

Proof. Let c be any configuration on $\Gamma(A)$. We can quickly see that the above mapping respects addition. Now define

$$\vec{1} \text{ the all ones configuration on } \Gamma(A),$$

$$c_{\text{big}} = c_{\max} + \vec{1},$$
$$c_{\text{zero}} = c_{\text{big}} - (c_{\text{big}})^\circ.$$

We then have $c_{\text{zero}} = c_{\text{big}} - (c_{\text{big}} - A\sigma)$ for some legal firing script σ, so $c_{\text{zero}} = 0 \bmod \mathcal{A}$. Furthermore, since c_{big} is unstable at every vertex, $c_{\text{zero}} \geq \vec{1}$.

We first show that if c is recurrent $(c + c_{\text{zero}})^\circ = c$. Let c be recurrent; then $(c_{\text{big}} + a) \xrightarrow{\tau} c$ for some configuration a and legal firing script τ. And let σ be a legal firing script stabilizing c_{big}. Then

$$\begin{aligned} c_{\text{big}} + a + c_{\text{zero}} &= c_{\text{big}} + a + c_{\text{big}} - c_{\text{big}}^\circ \\ &\xrightarrow{\sigma} a + c_{\text{big}} + c_{\text{big}}^\circ - c_{\text{big}}^\circ \\ &= a + c_{\text{big}} \\ &\xrightarrow{\tau} c. \end{aligned}$$

Clearly we also have $c_{\text{big}} + a + c_{\text{zero}} \xrightarrow{\tau} c + c_{\text{zero}}$. So by Corollary 1.9 $(c + c_{\text{zero}})^\circ = c$.

Let $k \gg 0$ such that $c + kc_{\text{zero}} \geq c_{\max}$, then we know

$$(c + kc_{\text{zero}})^\circ = c \bmod \mathcal{A}.$$

Then, since $(c + kc_{\text{zero}})^\circ = ((c + kc_{\text{zero}} - c_{\max}) + c_{\max})^\circ$, c must be recurrent by Theorem 1.26. So we know that the elements of $\mathbb{R}^n/\mathcal{A}$ are represented by recurrent configurations, hence the mapping is surjective. We would now like to show that these representatives are unique.

Let $c' = c'' \bmod \mathcal{A}$ where c' and c'' are recurrent configurations on $\Gamma(A)$. Then $c' = c'' - A\tau$ for some $\tau \in \mathbb{Z}^n$. Let

$$\tau_- = \begin{cases} \tau_i & \text{for } \tau_i < 0, \\ 0 & \text{else,} \end{cases}$$

$$\tau_+ = \begin{cases} \tau_i & \text{for } \tau_i > 0, \\ 0 & \text{else.} \end{cases}$$

Then define $c = c' + A\tau_+ = c'' + A\tau$ and let $k \gg 0$ such that

$$c + kc_{\text{zero}} \geq \sum_{v \in [n]} \tau_{+_v} A_{vv} v - \sum_{v \in [n]} \tau_{-_v} A_{vv} v.$$

Then, through legal firings,

$$c + kc_{\text{zero}} = c' + A\tau_+ + kc_{\text{zero}} \to c' + kc_{\text{zero}} \to c\,,'$$

and

$$c + kc_{\text{zero}} = c'' + A\tau_- + kc_{\text{zero}} \to c'' + kc_{\text{zero}} \to c''.$$

Thus by Corollary 1.9 we have that $c' = c''$. $\square$

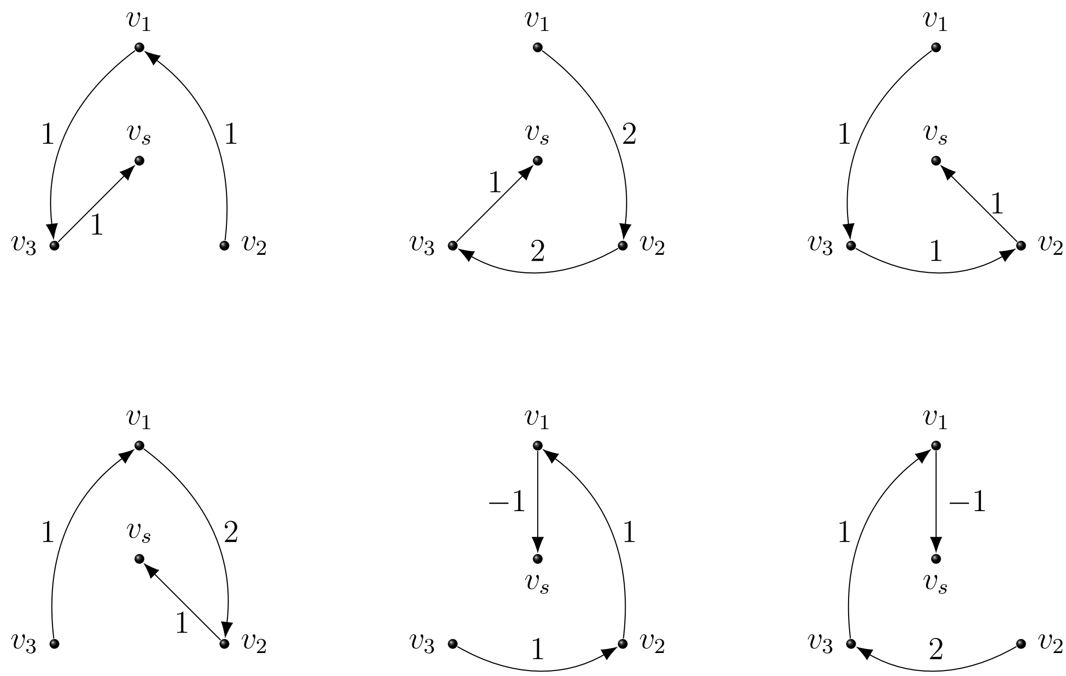


Figure 1.3: Spanning trees of $\Gamma(A)$ in Example 1.34

A *spanning tree* of a directed graph $\Gamma(A)$ *rooted* at v_s is a directed subgraph that contains $[n]$, and each $v \in [n]$ has one outgoing edge while v_s has no outgoing edges. The *weight* of a spanning tree is the product of its edge weights, where the weight of an edge is the same as for $\Gamma(A)$.

Theorem 1.32. *Let A be a nonsingular integer M-matrix and $\Gamma(A)$ its graph. Then the determinant of A equals the sum of the weights of all spanning trees directed into the sink vertex.*

Proof. The above Theorem follows directly from Kirchhoff's Matrix tree theorem. □

Corollary 1.33. *If A is an integer M-matrix, the order of $\mathcal{S}(A) \cap \mathbb{Z}^n$ is equal to the determinant of A and the sum of the weights of the spanning trees on $\Gamma(A)$.*

Proof. This follows directly from Theorem 1.32 and Theorem 1.30. □

Example 1.34. Consider

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

In Figure 1.3 we see the spanning trees of $\Gamma(A)$. Starting in the top left corner and traversing across the weights of the trees are 1, 4, 1, 2, -1, -2 respectively. In accordance with Corollary 1.33 these weights sum to $\det(A) = 5$.

1.3.2 Burning Configuration

The *support* of a configuration c on $\Gamma(A)$, i.e., of a vector $c \in \mathbb{R}^n$ is

$$\operatorname{supp}(c) = \{v \in [n] : c_v \neq 0\}.$$

The *closure* of the support of c, denoted $\overline{\operatorname{supp}}(c)$, is the set of $j \in [n]$ such that there exists a directed path in $\Gamma(A)$ to j from some vertex $i \in \operatorname{supp}(c)$.

Definition 1.35. A configuration $\mathbf{b} \geq 0$ on $\Gamma(A)$ is called a *burning configuration* if

1. $\mathbf{b} = 0 \mod \mathcal{A}$,
2. $\overline{\operatorname{supp}}(\mathbf{b}) = [n]$.

Definition 1.36. If $\mathbf{b}$ is a burning configuration for A, then $\sigma_{\mathbf{b}} = A^{-1}\mathbf{b}$ is the *burning script.*

Remark 1.37. If $\mathbf{b}$ is a burning configuration on $\Gamma(A)$ and $k \gg 0$ then there exists a legal firing script σ such that $k\mathbf{b} \xrightarrow{\sigma} c$ with $c \geq c_{\max}$. This follows since $\mathbf{b} \geq 0$ and $\overline{\operatorname{supp}}(\mathbf{b}) = [n]$.

Theorem 1.38. *Let A be a Z-matrix, then the following are equivalent:*

1. *A is avalanche finite;*
2. *$\Gamma(A)$ has a burning configuration;*
3. *There exists a $\sigma > 0$ such that $A\sigma > 0$, i.e. A is* semi-positive.

Proof. ($1 \Rightarrow 2$) Let A be avalanche finite and $c = c_{\max} - A\vec{1}$ a configuration on $\Gamma(A)$. Let $\sigma \geq 0$ be the legal firing script stabilizing $c_{\max} - A\vec{1}$, that is

$$\left(c_{\max} - A\vec{1}\right)^{\circ} = c_{\max} - A\left(\vec{1} + \sigma\right).$$

Let $\mathbf{b} = A\left(\vec{1} + \sigma\right)$. Then $\mathbf{b} \geq 0$ and $\mathbf{b} \mod \mathcal{A} = 0$. Assume $\overline{\operatorname{supp}}(\mathbf{b}) \neq [n]$, and define $W = [n] \setminus \overline{\operatorname{supp}}(\mathbf{b})$. For any subset $X \subseteq [n]$, define A_X to be the submatrix of A formed from the rows and columns corresponding to elements of X. By permuting vertices, we may assume $W = [m]$ for some $m < n$. Hence, A has the form

$$A = \left(\begin{array}{c|c} A_W & 0 \\ \hline * & A_{\overline{\operatorname{supp}}(\mathbf{b})} \end{array}\right).$$

Let σ_W be the first m elements of σ. Since any path from W to $\overline{\operatorname{supp}}(\mathbf{b})$ now goes directly to the sink, we can see that A_W must be avalanche finite. Hence ${A_W}^{-1}$ exists. Since the elements in W are not in $\overline{\operatorname{supp}}(\mathbf{b})$ it must be the case that $\mathbf{b}_W = 0$. Then we have $A_W\sigma_W = 0$, which only occurs if $\sigma_W = 0$, but $\sigma_W \geq 1$. By contradiction $\overline{\operatorname{supp}}(\mathbf{b}) = [n]$ and $\Gamma(A)$ has a burning configuration.

$(2 \Rightarrow 3)$ Let $\mathbf{b}$ be a burning configuration with burning script $\sigma_{\mathbf{b}} = A^{-1}\mathbf{b} > 0$, then there exists a τ such that for all sufficiently large $k > 0$, τ is a legal firing script for $k\mathbf{b}$ such that $k\mathbf{b} \to c$ for some $c > 0$. That is

$$k\mathbf{b} - A\tau = A\left(k\sigma_{\mathbf{b}} - \tau\right) > 0.$$

Then since τ and k are independent of one another, for a large enough $k' > k$, we have $k'\sigma_{\mathbf{b}} - \tau > 0$. Let $\sigma = k'\sigma_{\mathbf{b}} - \tau$, then $\sigma > 0$ and $A\sigma > 0$.

$(3 \Rightarrow 1)$ Let A be a Z-matrix such that there exists a $\sigma > 0$ with $A\sigma > 0$. Let c be an unstable configuration on $\Gamma(A)$. Then there exists a $k > 0$ and legal firing script τ such that $k\sigma - A\tau = d > c$. Then

$$c' = c - A\left(k\sigma - \tau\right) < 0.$$

So by Theorem 1.7 there exists a legal firing script ϕ such that $\phi < (k\sigma - \tau)$ and $c - A\phi$ is stable. □

Remark 1.39. We construct the $\mathbf{b}$ and corresponding $\sigma_{\mathbf{b}}$ with a greedy algorithm. Start with $\sigma^1 = \vec{1}$ and $b^1 = A\sigma^1$. If there is a index i such that $b^1_i < 0$, define $\sigma^2 = \sigma^1 + e_i$ and $b^2 = A\sigma^2$. Continue in this manner until $b^j \geq 0$, then let $\mathbf{b} = b^j$. Note that this is the same burning configuration found in the proof of Theorem 1.38. The key idea is that for any configuration c, we have that $c_{\max} - c$ is unstable at vertex v if and only if $c_v < 0$.

Remark 1.40. Let $\mathbf{b}$ be the burning configuration constructed in Remark 1.39 and $\mathbf{b}'$ be any other burning configuration on $\Gamma(A)$. Then $\mathbf{b}$ is the minimal burning configuration on $\Gamma(A)$, that is $\mathbf{b} \geq \mathbf{b}'$. To see this we need to show $\sigma_{\mathbf{b}'} \geq \sigma_{\mathbf{b}}$. Let $\sigma_{\mathbf{b}} = \vec{1} + \sigma$ where σ is the legal firing script stabilizing $c_{\max} - A\vec{1}$. Consider $(c_{\max} - A\vec{1}) - A(\sigma_{\mathbf{b}'} - \vec{1}) = c_{\max} - A\sigma_{\mathbf{b}'}$. Since $\mathbf{b}' \geq 0$ this is stable, though the firings may not be legal. And $c_{\max} + A\vec{1} \xrightarrow{\sigma} (c_{\max} + A\vec{1})^\circ$ is a legal script, thus by the least action principle $\sigma \leq \sigma_{\mathbf{b}'} - \vec{1}$. That is $\sigma_{\mathbf{b}} \leq \sigma_{\mathbf{b}'}$ and $\mathbf{b} \leq \mathbf{b}'$.

Example 1.41. Let us find the burning configuration for

$$A = \begin{pmatrix} 3 & -1 & -3 \\ -1 & 4 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

We have $b^1 = (-1, 2, 1)$, so we set $\sigma^2 = (2, 1, 1)$. This gives $b^2 = (2, 1, 0) \geq 0$, so $\mathbf{b} = b^2$.

Theorem 1.42. *Let $\mathbf{b}$ be a burning configuration for a nonsingular M-matrix A, with burning script $\sigma_{\mathbf{b}}$. Let e be the identity of $\mathcal{S}(A)$. Then*

1. $(k\mathbf{b})^\circ = e$ *for some* $k \gg 0$.
2. *A configuration* $c \geq 0$ *is recurrent if and only if* $(\mathbf{b} + c)^\circ = c$.

3. *A configuration* $c \geq 0$ *is recurrent if and only if* $(\mathbf{b}+c) \xrightarrow{\sigma_{\mathbf{b}}} (\mathbf{b}+c)^{\circ}$.

4. $\sigma_{\mathbf{b}} \geq \vec{1}$.

5. *Suppose* c *is a stable configuration and* σ *a legal firing script such that* $(\mathbf{b}+c) \xrightarrow{\sigma} (\mathbf{b}+c)^{\circ}$. *Then* $\sigma \leq \sigma_{\mathbf{b}}$.

Proof.

1. For large enough k, by repeatedly firing $i \in \operatorname{supp}(\mathbf{b})$, and legally firing the corresponding path to elements not in $\operatorname{supp}(\mathbf{b})$, we can obtain $k\mathbf{b} \to c + c_{\max}$ for some configuration $c \geq 0$, so $(k\mathbf{b})^{\circ}$ is recurrent. Since $\mathbf{b}$ is a burning configuration, we have
$$k\mathbf{b} \bmod \mathcal{A} = k(\mathbf{b} \bmod \mathcal{A}) = 0 \bmod \mathcal{A}.$$
Then since the mapping $\mathcal{S}(A) \to \mathbb{R}^n/\mathcal{A}$ is an isomorphism, $(k\mathbf{b})^{\circ}$ must be the identity element of $\mathcal{S}(A)$.

2. ($\Rightarrow$) Suppose $c \geq 0$ is a recurrent configuration, then $(\mathbf{b}+c)^{\circ}$ is recurrent as well. Then we must have $(\mathbf{b}+c)^{\circ} = c$ since $\mathbf{b} = 0 \bmod A$.
($\Leftarrow$) Suppose $(\mathbf{b}+c)^{\circ} = c$. Fix an integer $k \gg 0$ such that the stabilization of $k\mathbf{b}$ is e. Then $c = (k\mathbf{b}+c)^{\circ} \to (e+c)^{\circ}$. Since e is recurrent c must be so as well.

3. Let $c \geq 0$ be a configuration and σ be the legal firing script stabilizing $\mathbf{b}+c$. By part (2) c is recurrent if and only if $c = (c+\mathbf{b})^{\circ} = \mathbf{b} + c - A\sigma$. Whence, $A\sigma = \mathbf{b}$. So $\sigma = A^{-1}\mathbf{b} = \sigma_{\mathbf{b}}$.

4. This follows directly from construction of the minimal burning configuration and the fact that A^{-1} is nonnegative. Independent of the minimal burning configuration we can show this as follows; since $c_{\max}$ is recurrent we have $(\mathbf{b}+c_{\max})^{\circ} \xrightarrow{\sigma_{\mathbf{b}}} c_{\max}$. Let $j \in [n]$. Since $\overline{\operatorname{supp}}(\mathbf{b}) = [n]$ there exists an $v_i \in \operatorname{supp}(\mathbf{b})$ and a path $v_i, v_{i_1}, v_{i_2}, \ldots, v_j$ so that $v_i, v_{i_1}, v_{i_2}, \ldots, v_j$ is a legal firing sequence for $\mathbf{b}+c_{\max}$. This sequence is legal since $(c_{\max}+\mathbf{b})_{v_i} > c_{\max v_i}$, and firing v_i causes v_{i_1} to be unstable were it not already. This process continues until hitting v_j. Then, since v_j is arbitrary, every vertex must fire at least once and $\sigma_{\mathbf{b}} \geq \vec{1}$.

5. Suppose c is a stable configuration and σ is the legal firing script stabilizing $\mathbf{b}+c$. Then $c \leq c_{\max}$ so σ is also legal for $c_{\max}+\mathbf{b}$. Since $c_{\max}$ is recurrent by (3), we must have $\sigma_{\mathbf{b}} \geq \sigma$.

$\square$

1.3.3 Superstables and Duality

Definition 1.43. We say a configuration c on $\Gamma(A)$ is *superstable* if there exists no $\sigma \gneq 0$ such that $c - A\sigma \geq 0$.

The following example demonstrates that c being a stable configuration does not necessarily mean that c is superstable.

Example 1.44. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

Consider the stable configuration $(1,1,1)$ on $\Gamma(A)$. We check if there is a $\tau \gneq 0$ such that $A\tau \leq (1,1,1)$. Note that $\tau = (1,1,1)$ gives

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so $(1,1,1)$ is not a superstable configuration.

There is an interesting duality between superstables and recurrent configurations.

Theorem 1.45. *c is a recurrent configuration on $\Gamma(A)$ if and only if $c_{\max} - c$ is a superstable configuration.*

Proof. ($\Rightarrow$) Let A be an M-matrix with recurrent configuration c, and say $c_{\max} - c$ is not a superstable configuration. Then there exists a $\sigma \geq 0$ such that $c_{\max} - c - A\sigma \geq 0$. Since c is recurrent there exists some $c_{\text{large}} > c_{\max}$ such that $(c_{\text{large}})^{\circ} = c$ by some legal firing script σ'. Rewriting we get

$$c_{\max} - c - A\sigma = c_{\max} - (c_{\text{large}} - A\sigma') - A\sigma = c_{\max} - c_{\text{large}} - A(\sigma - \sigma') \geq 0.$$

Which clearly implies $A(\sigma - \sigma') \leq 0$. Hence by Remark 1.8, we have $\sigma \leq \sigma'$. Furthermore, $c_{\max} \geq c_{\text{large}} - A(-\sigma + \sigma')$, so $(-\sigma + \sigma')$ is a legal firing script terminating in a stable configuration; by construction we know $-\sigma + \sigma' \leq \sigma'$, but that says $\sigma \leq 0$. By contradiction we have $c_{\max} - c$ is a superstable configuration.

($\Leftarrow$) Let $c_{\max} - c$ be a superstable configuration and c not be a recurrent configuration. Furthermore, let $\sigma \geq 0$. Then for at least one vertex $v \in [n]$ we have

$$(c_{\max} - c - A\sigma)_v < 0 \Rightarrow (c_{\max})_v < (c + A\sigma)_v.$$

So $c + A\sigma$ must be unstable. Let $\mathbf{b}$ be the burning configuration for A, and σ the legal firing script stabilizing $(c + \mathbf{b})$. Define $\tau = \sigma_{\mathbf{b}} - \sigma$, then $\tau \geq 0$, and by Theorem 1.42 we have $\tau \leq \sigma_{\mathbf{b}}$, so $c + A\tau$ is unstable. But we also have

$$(c + \mathbf{b})^{\circ} = c + A\sigma_{\mathbf{b}} - A\sigma = c + A\tau,$$

which is stable. Thus c must be recurrent. ☐

Example 1.46. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{pmatrix}.$$

Earlier we showed that $(0,3,1)$ is a recurrent configuration on $\Gamma(A)$. Theorem 1.45 tells us that $(1,3,2)-(0,3,1)=(1,0,1)$ is a superstable configuration. Note that we now have a quicker method to determining if a configuration is superstable. Taking our earlier non-superstable configuration $(1,1,1)$, we observe that there is no element c in the set of recurrents such that $c_{\max} - c = (1,1,1)$.

Chapter 2

Symmetric Configurations

Florescu, et. al. [7] showed that when considering graphs with at least one axis of symmetry the reduced Laplacian could be modified in such a way as to only consider the configurations and firings on a representative fraction of the graph. However, the modified matrix that is obtained is not always the reduced Laplacian of a graph; thankfully, we observe that the augmented matrix is always an M-matrix. Hence, from section 1.1 of this thesis, we now have a beneficial way of visualizing these representations. The propositions in this section follow directly from Florescu, et. al., [7]. The proofs are omitted as they straightforwardly extend to nonsingular real M-matrices. For this section, let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix.

Definition 2.1. Let G be a finite group. An *action* of G on $\Gamma(A)$ is an action on $[n] \cup \{s\}$. In detail, we have a function

$$G \times ([n] \cup \{s\}) \longrightarrow [n] \cup \{s\},$$

with the following properties:

1. For the identity e of G, $ev = v$ for all $v \in [n] \cup s$;
2. The actions G is associative, that is $(gh)v = g(hv)$ for $g, h \in G$ and $v \in [n]\cup\{s\}$;
3. If $g \in G$, then $gs = s$, that is the sink is fixed by G;
4. Edge weights of $\Gamma(A)$ are preserved under G, that is for $u, v \in [n] \cup \{s\}$ and $g \in G$ we have $\mathrm{wt}(u, v) = \mathrm{wt}(gu, gv)$.

From now on, let G be a finite graph acting on $\Gamma(A)$. By linearity the action extends to an action on $\mathbb{R}^n$ after identifying each vertex i with the standard basis vector e_i. Hence, G acts on configurations on $\Gamma(A)$. We say a configuration $c \in \mathbb{R}^n$ on $\Gamma(A)$ is *symmetric* if for every $g \in G$ and $v \in [n]$, we have $c_v = (gc)_v$.

Proposition 2.2. The action G is commutes with stabilization. That is for a configuration c on $\Gamma(A)$, $g(c^\circ) = (gc)^\circ$.

Definition 2.3. The set *orbits* of $\Gamma(A)$ under G are $\mathcal{O} = \{Gv : v \in [n]\}$.

The *symmetrization* under G, denoted A^G, is the matrix obtained as follows.

Definition 2.4. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of $\Gamma(A)$ under G and choose $\omega_i \in \mathcal{O}_i$ for $i \in [m]$. So $\omega_i \in [n]$ for each i. Let the $n \times m$ matrix K be defined by $Ke_j = \sum_{k \in \mathcal{O}_j} Ae_k$. Then define the $m \times m$ matrix A^G whose i-th row is the ω_i-th row of K.

Let $c \in \mathbb{R}^n$ be a symmetric configuration on $\Gamma(A)$ such that c_v is unstable. Then since c is symmetric it must be the case that every vertex in the orbit of v is unstable, that is $(gc)_v$ is unstable for all $g \in G$. By firing gv we represent firing all vertices in the orbit of v, which we know to be legal for $\Gamma(A)$. We have the mappings,

$$(\mathbb{R}^n)^G \to \mathbb{R}^m \qquad \text{and} \qquad \mathbb{R}^m \to (\mathbb{R}^n)^G$$
$$a \mapsto (a_{\omega_i}) \qquad\qquad\qquad b \mapsto b^G$$

where a is a symmetric configuration on $\Gamma(A)$ and b a configuration on $\Gamma(A^G)$ where $b_i^G = b_{\omega_i}$ if $v \in \mathcal{O}_{\omega_i}$.

Remark 2.5. Since A^G is obtained by summing the columns of A that correspond to distinct orbits of $\Gamma(A)$ and then picking the rows that corresponds to the chosen orbit representatives, it is clear that A having nonpositive off-diagonals implies that A^G will as well. Note that since $g(c^\circ) = (gc)^\circ$, if A is avalanche finite A^G must be so as well. Hence, A^G is a nonsingular M-matrix.

Corollary 2.6. *If c is recurrent on $\Gamma(A)$ then gc is recurrent on $\Gamma(A^G)$.*

Corollary 2.7. *If c is a symmetric configuration then its stabilization is symmetric as well.*

Proposition 2.8. The set of symmetric recurrent configurations $\mathcal{S}(A)^G$ form a subgroup of $\mathcal{S}(A)$.

Proposition 2.9. Let $\omega_1, \omega_2, \ldots, \omega_m$ be representatives of the orbits of $\Gamma(A)$ under G. Then

$$\mathcal{S}(A)^G \to \mathbb{R}^m / \mathrm{Im}_{\mathbb{Z}}(A^G)$$
$$c \mapsto (c_{\omega_i})_{i=1,\ldots,m}$$

is an isomorphism; additionally, if A is an integer matrix,

$$\mathcal{S}(A)^G \cap \mathbb{Z}^m \to \mathbb{Z}^m / \mathrm{Im}_{\mathbb{Z}}(A^G)$$
$$c \mapsto (c_{\omega_i})_{i=1,\ldots,m}$$

is an isomorphism.

Corollary 2.10. *If A is an integer matrix then the number of integer symmetric recurrents on $\Gamma(A)$ is* $\det(A^G)$.

Example 2.11. Let

$$A = \begin{pmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

Note that $\det(A) = 11$. So there are eleven recurrent configurations on $\Gamma(A)$: $(2,2,1,1)$, $(2,2,0,1)$, $(1,2,1,0)$, $(2,0,1,1)$, $(0,2,1,1)$, $(2,1,1,0)$, $(1,2,0,1)$, $(2,1,0,1)$, $(2,2,1,0)$, $(1,2,1,1)$, $(2,1,1,1)$. Figure 2.1 shows $\Gamma(A)$ with its axis of symmetry in red. Let $G \approx \mathbb{Z}/2\mathbb{Z}$, be the group generated by flipping around this axis. The symmetrization of A is

$$A^G = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Consider the unstable symmetric configuration $c = (2,2,2,2)$, then the corresponding element on $\Gamma(A^G)$ is $c^G = (2,2)$. The stabilization process is as follows.

$$(2,2,2,2) \xrightarrow{v_3,v_4} (3,3,1,1) \xrightarrow{v_1,v_2} (1,1,2,2) \xrightarrow{v_3,v_4} (2,2,1,1) = c_{\max}$$

$$(2,2) \xrightarrow{gv_3} (3,1) \xrightarrow{gv_1} (1,2) \xrightarrow{gv_3} (2,1) = gc_{\max}$$

Since $\det(A^G) = 1$, we know that $\Gamma(A)$ has one symmetric recurrent configuration. Specifically, $c_{\max}$ is the only symmetric recurrent configuration on $\Gamma(A)$.

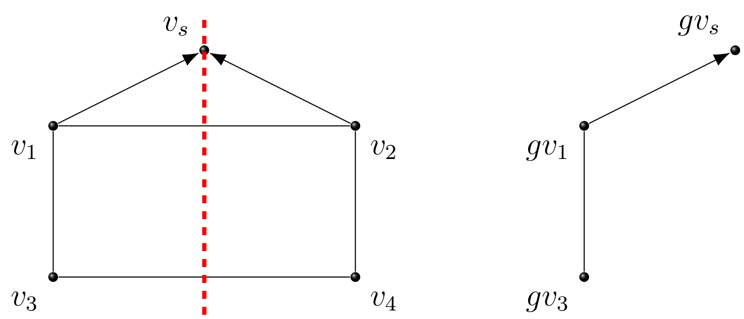


Figure 2.1: $\Gamma(A)$ and $\Gamma(A^G)$ for Example 2.11.

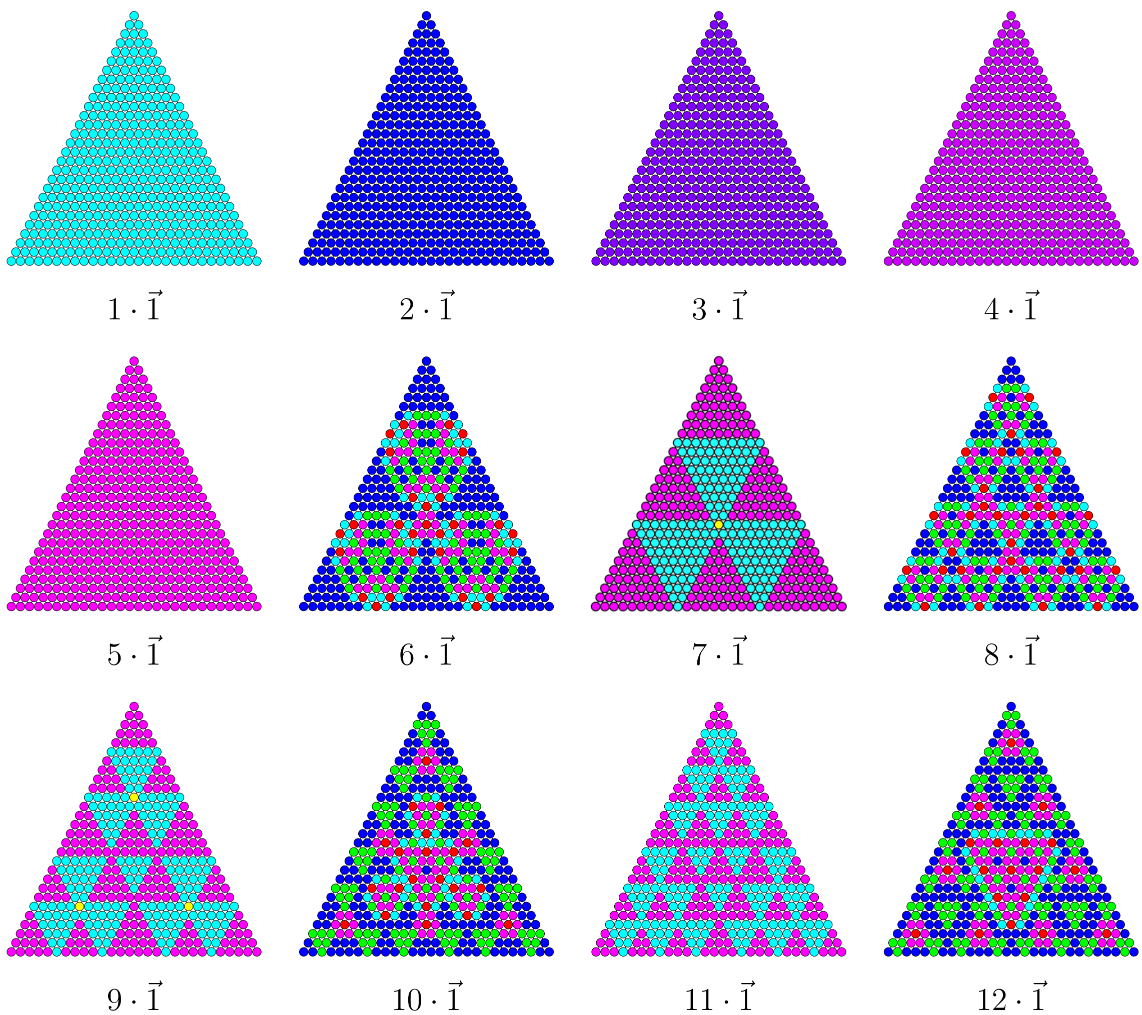
2.1 Hexagonal Lattice with Triangular Boundary

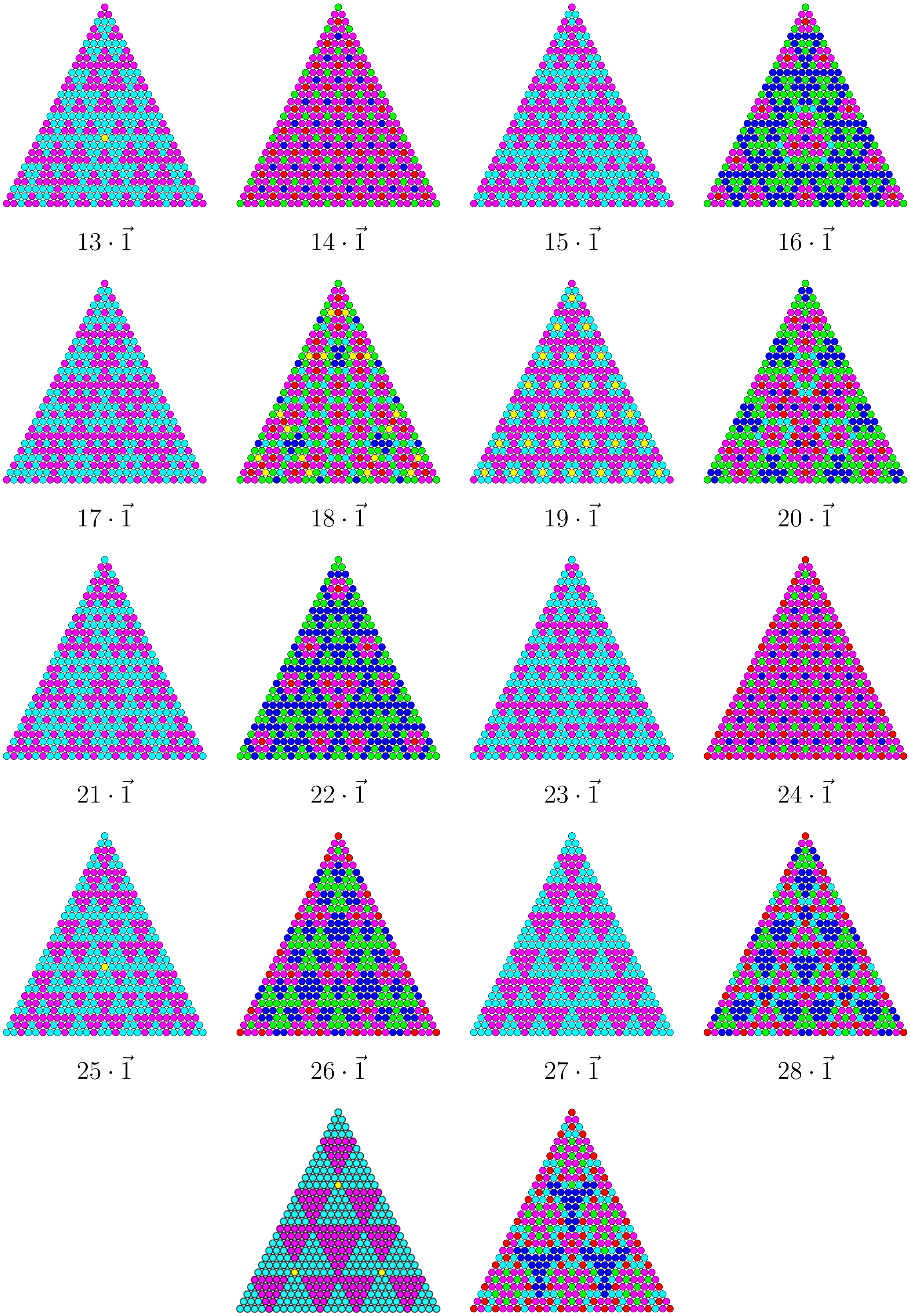
We turn now to the original motivation for this thesis: the stabilization of the all-ones configuration on the hexagonal lattice with triangular boundary, T_n. Each vertex of T_n has an outdegree of 6 and there are $\sum_{k=1}^{n} k = \binom{n+1}{2}$ vertices. Consider Figure 2.1, we have an equilateral triangle with five vertices on each side; the three corner vertices have a path to the sink of weight four (represented by four dashed lines), while all other vertices on the edge have an edge of weight two to the sink. In general T_n will have $3(n-2)$ vertices with edges to the sink of weight two and 3 vertices with edges to the sink of weight 4. The reduced Laplacian A for T_n is a nonsingular M-matrix. Throughout this section, we restrict our attention to integer configurations. The *order* of an integer configuration c on T_n, denoted order(c), is the smallest possible integer $k \geq 1$ such that there exists a $\sigma \in \mathbb{Z}^n_{\geq 0}$ with $kc = A\sigma$. Additionally, if c is recurrent, order(c) is the first stabilization of an integer multiple of c for which we see the identity of $\mathcal{S}(A)$. Furthermore, k must divide the order of $\mathcal{S}(A) \cap \mathbb{Z}^n$ and $\mathcal{S}(A)^G \cap \mathbb{Z}^n$, that is k must divide $\det(A)$ and $\det(A^G)$.

Conjecture 2.12. The order of the all-ones configuration on T_n is:

$$\text{order}(\vec{1}) = \begin{cases} 2n+4 & n \text{ odd,} \\ n+2 & n \text{ even.} \end{cases}$$

Figure 2.2 shows the stabilization of $k \cdot \vec{1}$ on T_{28} for $k = 1, 2, \ldots, 30$. We observe that for $k = 30$ we see the identity element of $\mathcal{S}(A)$, hence order$(\vec{1}) = 30$ as expected.





$13 \cdot \vec{1}$ $14 \cdot \vec{1}$ $15 \cdot \vec{1}$ $16 \cdot \vec{1}$

$17 \cdot \vec{1}$ $18 \cdot \vec{1}$ $19 \cdot \vec{1}$ $20 \cdot \vec{1}$

$21 \cdot \vec{1}$ $22 \cdot \vec{1}$ $23 \cdot \vec{1}$ $24 \cdot \vec{1}$

$25 \cdot \vec{1}$ $26 \cdot \vec{1}$ $27 \cdot \vec{1}$ $28 \cdot \vec{1}$

$29 \cdot \vec{1}$ $30 \cdot \vec{1}$

Figure 2.2: Stabilization of $k \cdot \vec{1}$ on T_{28} for $k = 1, 2, \ldots, 30$.

We now walk through the steps of showing $\vec{1}$ on T_5 has order 14. Since T_5 has 15 vertices excluding the sink, and $\vec{1}$ is symmetric, we will work A^G, where A is the reduced Laplacian of T_5.

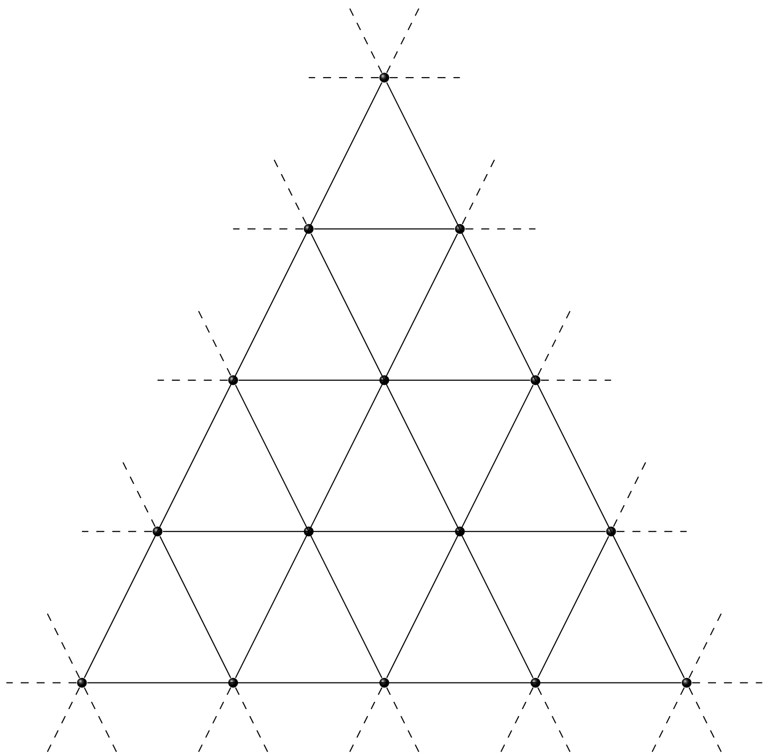


Figure 2.3: T_5: Hexagonal Lattice with Boundary of 5.

$$A^G = \begin{array}{c} \\ gv_1 \\ gv_4 \\ gv_5 \\ gv_6 \end{array} \begin{array}{c} \begin{array}{cccc} gv_1 & gv_4 & gv_5 & gv_6 \end{array} \\ \begin{pmatrix} 4 & 0 & -2 & -2 \\ 0 & 6 & -2 & 0 \\ -1 & -1 & 5 & -1 \\ -2 & 0 & -2 & 6 \end{pmatrix} \end{array}.$$

Conjecture 2.12 tells us that $\vec{14}$ should be in the image of A^G. Consider

$$\sigma = \begin{pmatrix} 12 \\ 5 \\ 8 \\ 9 \end{pmatrix}.$$

Then $A^G\sigma = \vec{14}$. The determinant of A^G is 392 which has prime factorization $2^3 \times 7^2$. Since A is an M-matrix, quickly computing $(A^G)^{-1}(k\vec{2})$ for $k = 2, 4, 7, 8$ gives non-integer σ's. Hence order$(\vec{1}) = 14$.

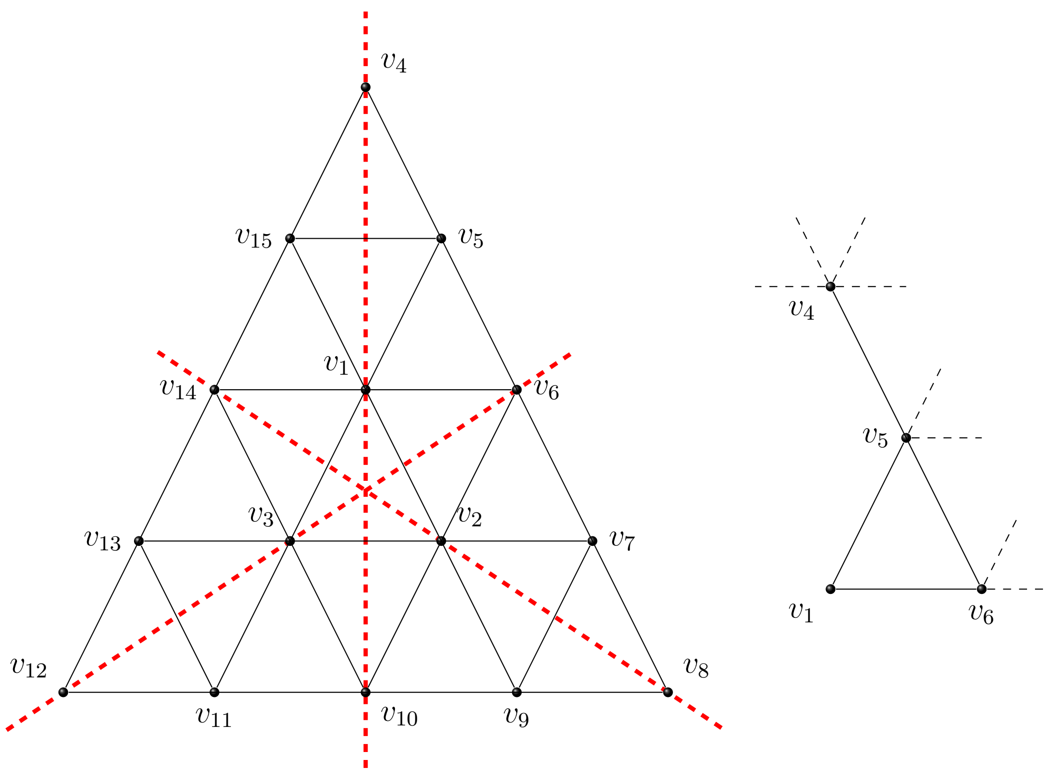


Figure 2.4: Symmetries of T_5.

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