Rowmotion over P-partitions

A Thesis
Presented to
The Division of Mathematical and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

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September 2021

Approved for the Division
(Mathematics)

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Acknowledgements

To Dave, thank you for guiding me throughout the year.
To my parents, thank you for supporting me under any circumstances.
To Rodger, thank you for being there with me, and I'll always be there with you.

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Abstract

This thesis will mainly focus on a conjecture raised by Hopkins (2020a). He conjectures that if two posets have the same comparability graph, then there exists an bijection between their P-partition orbits under piecewise-linear rowmotion action which preserves orbit length and total down-degree. Chapter 1 introduces the background of posets, comparability graphs, and piecewise rowmotion. Chapter 2 contains a new proof of Hopkins' conjecture of P-partitions at height 1. Hopkins has proved this case earlier in terms of order ideals. Chapter 2 also provides a proof of the conjecture under the special case when the two posets are dual to each other. Then we construct a bijection between P-partitions of any pair of posets with the same comparability graph. The bijection does not commute with rowmotion, but it leads to a proof of the conjecture in cases more general than those above.

Introduction

A **partially ordered set (or poset)** P is a set equipped with a binary relation $\leq$ that is reflexive, antisymmetric and transitive. An **order ideal** of a poset P is a subset $I \subseteq P$ such that for all $x \in I$ and $y \in P$, if $y \leq x$, then $y \in I$. The set of all order ideals is called $J(P)$. **Rowmotion** is an invertible operator acting on order ideals, defined as $\mathrm{row}(I) = \{y \in P : y \leq x \text{ for some minimal elements } x \text{ in } P \setminus I\}$. This operator has been studied for many decades due to its nice behavior. See Hopkins (2020b) for references.

A weakly order-preserving map $f : P \to \{0, \cdots, \ell\}$ is called a **P-partition of height** ℓ, and $PP^{\ell}(P)$ is the set of all P-partitions. The idea of P-partitions generalizes the concept of order ideals, which is given by the special case of P-partitions when $\ell = 1$. See Gessel (2016) for some history of P-partitions. Rowmotion over order ideals is also generalized to P-partitions by Einstein & Propp (2021). The **(piecewise-linear) rowmotion** over a poset P is defined as the composition of **(picewise-linear) toggles** at its posets element, defined as $\tau_p : PP^{\ell}(P) \to PP^{\ell}(P)$,

$$\tau_p(f)(x) = \begin{cases} f(x) & \text{if } p \neq x, \\ \min\{f(y) : y \gtrdot p\} + \max\{f(y) : p \gtrdot y\} - f(p) & \text{if } p = x, \end{cases}$$

where $\min(\varnothing) = \ell$ and $\max(\varnothing) = 0$.

Down-degree of a P-partition is the sum over i over the maximal elements of the preimage $f^{-1}(\{0, \cdots, i\})$. The **comparability graph** $\mathrm{com}(P)$ of a poset P is an undirected graph where $x, y \in P$ are connected with an edge if and only if x and y are comparable. Certainly the comparability graph cannot decide a poset, however, posets that share the same comparability graphs also share some interesting properties. Hopkins (2020a) raised the following conjecture:

Conjecture 2.0.1. [Hopkins (2020a) Conjecture 4.38] Let P and Q be posets such that $\mathrm{com}(P) \simeq \mathrm{com}(Q)$. Then there exists a bijection φ between the row orbits of $PP^{\ell}(P)$ and the row orbits of $PP^{\ell}(Q)$ such that for all $\mathcal{O} \subseteq PP^{\ell}(P)$, we have

1. $|\mathcal{O}| = |\varphi(\mathcal{O})|$.

2. $\mathrm{ddeg}(\mathcal{O}) = \mathrm{ddeg}(\varphi(\mathcal{O}))$.

where $\mathrm{ddeg}(\mathcal{O})$ is the sum of the down-degree of all P-partitions in $\mathcal{O}$.

Chapter 1 begins with an induction about posets, P-partitions and their down-degree. Then we look into posets with the same comparability graph, understanding that if $\mathrm{com}(P) \simeq \mathrm{com}(Q)$, then Q must be obtained from P by dualizing a sequence of autonomous subsets. Finally we learn about the piecewise-linear rowmotion acting on P-partitions and how this rowmotion action relates to the down-degree of the P-partitions. We also see that P-partition of height 1 can be viewed as order ideals.

Chapter 2 mainly focuses on the Conjecture 2.0.1 raised by Sam Hopkins. It has already been proved by Hopkins for order ideals $J(P)$. Chapter 2 starts with Theorem 2.1.1, which rephrases Hopkins' proof in the language of P-partitions of height 1, as a special case inside Conjecture 2.0.1. Then we construct a *complement* map between P-partitions, which inversely commutes with the rowmotion operator, and prove Theorem 2.2.1, which establishes the conjecture under another special case when the two posets are dual to each other. Finally we extend the constructed complement map to arbitrary posets P and Q, where Q is obtained from P by dualizing some autonomous subset A, and P contains no incomparable element with A. The new complement map does not commute with the rowmotion operator, but we conjecture (Conjecture 2.2.4) that it behaves nicely on preserving the size and the total down-degree of the rowmotion orbit generated by the corresponding P-partitions. This conjecture holds under our earlier Theorems 2.1.1 and 2.2.1. We then use Hall Marriage Theorem to prove Theorem 2.2.6, which is Hopkins' conjecture under the conditions of Conjecture 2.2.4.

Chapter 1

Poset, Comparability Graph and Rowmotion

1.1 Posets, P-partition and Down Degree

Definition 1.1.1. A **partially ordered set** (or **poset**) is a set P equipped with a binary relation $\leq$, which satisfies the following axioms:

1. Reflexivity: for all $a \in P$, $a \leq a$.

2. Antisymmetry: for all $a, b \in P$, if $a \leq b$ and $b \leq a$, then $a = b$.

3. Transitivity: for all $a, b, c \in P$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Let P be a poset, and we review some basic concepts. For $a, b \in P$, if $a \leq b$ or $b \leq a$, then we say a and b are **comparable**, otherwise **incomparable**.

If $a \leq b$ and $a \neq b$, then we write $a < b$ and we say that a is **strictly less than** b. Assume $a < b$ and there is no element $c \in P$ such that $a < c < b$, then we say a is **covered** by b, denoted as $b \gtrdot a$.

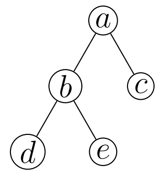
A **chain** is a totally ordered subset of P, under which every pair of elements are comparable. An **antichain** is a subset of P such that any pair of elements in the subset is incomparable.

A poset can have an infinite number of elements. However in this thesis, we are only looking at finite posets.

Definition 1.1.2. A **linear extension** of a poset is an ordering of all the elements $p_1, \cdots, p_n \in P$ such that $p_i < p_j$ implies $i < j$.

Example 1.1.3. The following picture is the **Hasse diagram** of the partially ordered set $P = \{a, b, c, d, e\}$ with covering relations $a \gtrdot b$, $a \gtrdot c$, $b \gtrdot d$. and $b \gtrdot e$.

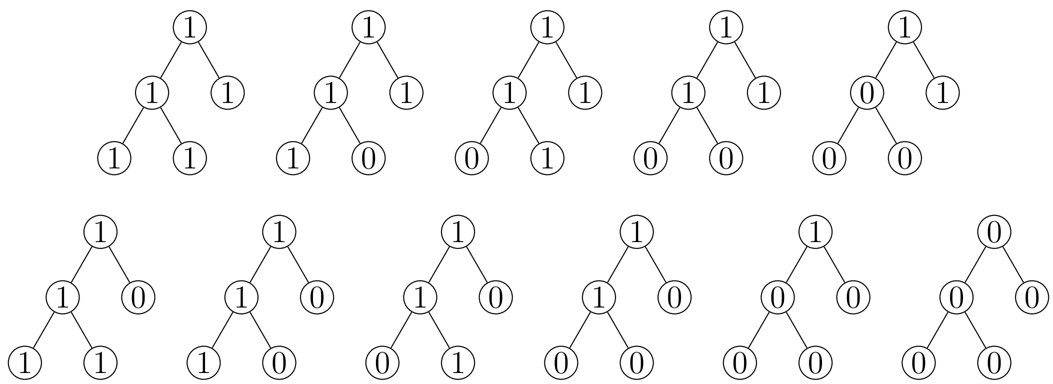
The permutations (e, d, c, b, a) and (e, d, b, c, a) are two possible linear extensions.



Definition 1.1.4. Let P be a finite poset and ℓ be a positive integer. Let $f\colon P \to \{0, \cdots, \ell\}$ be a function such that for all $a, b \in P$, $a \leq b$ implies $f(a) \leq f(b)$. Then f is **weakly order-preserving**, and is called a **P-partition of height** ℓ.

We let $PP^{\ell}(P)$ denote the set of all P-partitions of height ℓ.

Example 1.1.5. Consider the P-partition of poset P in Example 1.1.3 of height 1. Then $PP^1(P)$ consist of the following elements:



where the nodes of the Hasse diagrams are numbered by the values assigned by a P-partition. In this case, $|PP^1(P)| = 11$.

Definition 1.1.6. An **order ideal** of a poset P is a subset $I \subseteq P$ such that for all $x \in I$ and $y \in P$, if $y \leq x$, then $y \in I$.

We let $J(P)$ denote the set of all order ideals of P.

Observation 1.1.7. Let P be an arbitary poset, then there exists a bijection

$$\begin{gathered} \alpha : J(P) \to PP^1(P) \\ I \mapsto f_I \end{gathered}$$

where

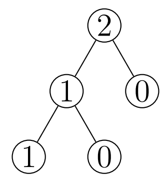
$$f_I(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I. \end{cases}$$

The fact that α is a bijection can be easily checked since P-partitions are weakly order-preserving. For instance, the P-partition defined in Example 1.1.3 has 11 order ideals corresponding to the 11 P-partitions listed in Example 1.1.5.

Definition 1.1.8. Let P be a poset, then the **down-degree** of P is the number of maximal elements of P, denoted as $\mathrm{ddeg}(P)$. We define the **down-degree** of the P-partition $f \in PP^{\ell}(P)$ as

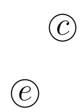
$$\mathrm{ddeg}(f) = \sum_{i=0}^{\ell-1} \mathrm{ddeg}(f^{-1}\{0, \cdots, i\}).$$

Example 1.1.9. Consider the poset P in previous example and let $f \in PP^2(P)$ with the following diagram

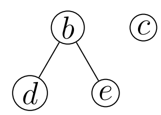


Then we know that $\mathrm{ddeg}(f) = \mathrm{ddeg}(f^{-1}\{0\}) + \mathrm{ddeg}(f^{-1}\{0,1\})$ by definition of down-degree.

Considering the two sub-posets separately, we see that $f^{-1}\{0\}$ and $f^{-1}\{0,1\}$ are sub-posets shown below:



$f^{-1}\{0\}$



$f^{-1}\{0,1\}$

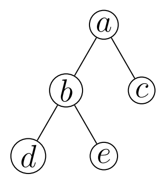
For $f^{-1}\{0\}$, we may see that c and e are both maximal elements, since none of them are covered by any elements. So $\mathrm{ddeg}(f^{-1}\{0\}) = 2$. Similarly for $f^{-1}\{0,1\}$, the maximal elements are b and c. The figure gives us that $\mathrm{ddeg}(f^{-1}\{0,1\}) = 2$ as well. Therefore, $\mathrm{ddeg}(f) = 2 + 2 = 4$.

Remark 1.1.10. Suppose $\ell = 1$ and $I \in J(P)$. Let α be the bijection given in Observation 1.1.7, then $\mathrm{ddeg}(I) = \mathrm{ddeg}(\alpha(I))$.

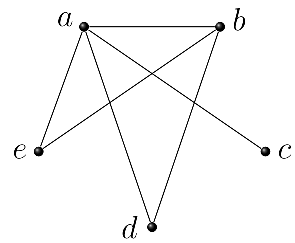
1.2 Comparability graph

Definition 1.2.1. Let P be a poset. The **comparability graph** of a poset is an undirected simple graph whose vertices are the elements of P. Let $p_1, p_2 \in P$, then p_1, p_2 are joined by an edge in the comparability graph if and only if p_1 and p_2 are comparable. We denote the comparability graph of P as $\mathrm{com}(P)$.

Example 1.2.2. Keep considering the poset P in Example 1.1.3. Then the comparability graph of P is



P



Comparability graph of P

Comparing with the Hasse diagram of the original poset, we see that two extra edges are added to the graph since a, d and a, e are comparable elements in the poset.

Since the comparability graph is undirected and simple, it is impossible for us to determine any poset within it. However, we can characterize posets sharing the same comparability graph.

Definition 1.2.3. Let P, Q be posets with the same comparability graph. Then we write $\mathrm{com}(P) \simeq \mathrm{com}(Q)$.

Definition 1.2.4. Let P be a poset. Let $A \subseteq P$ be a subset of P satisfying the following properties:

1. For all $a_1, a_2 \in A$ and for all $b \in P \setminus A$, we have $a_1 \leq b$ if and only if $a_2 \leq b$.

2. For all $a_1, a_2 \in A$ and for all $b \in P \setminus A$, we have $a_1 \geq b$ if and only if $a_2 \geq b$.

Then all the elements in A have the same order relations with all elements in $P \setminus A$, and we say A is **autonomous**.

A poset Q is said to be **obtained from** P **by dualizing** A if Q has the following properties:

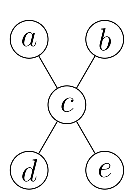
1. P and Q are the same as sets.

2. For $x, y \in Q \setminus A$, we have $x \leq_Q y$ if and only if $x \leq_P y$.

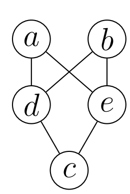
3. For $x, y \in A$, we have $x \leq_Q y$ if and only if $y \leq_P x$.

4. For $x \in A$ and $y \in Q \setminus A$, $x \leq_Q y$ if and only if $x \leq_P y$ and $y \leq_Q x$ if and only if $y \leq_P x$.

Example 1.2.5. Consider the following two posets P and Q.



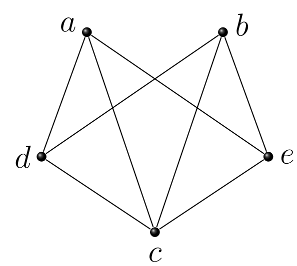
P



Q

Then $A = \{c, d, e\}$ is an autonomous poset of P, and Q is obtained from P by dualizing A, i.e., Q is obtained from P by simply reversing the order of the elements in A.

Their shared comparability graph is



Theorem 1.2.6 (Hopkins (2020a) Remark 4.19)**.** Let P, Q be posets, then $\mathrm{com}(P) \simeq \mathrm{com}(Q)$ if and only if there is a sequence of posets

$$P = P_0, P_1, \cdots, P_k = Q$$

such that P_i is obtained by dualizing an autonomous subset from P_{i-1} for all $1 \leq i \leq k$.

1.3 Piecewise rowmotion

Definition 1.3.1. Let P be a poset, and let $\hat{P}$ be a new poset obtained from P by adding a minimal element $\hat{0}$ and a maximal element $\hat{1}$. Let $f \in PP^{\ell}(P)$ be a P-partition. Then we view f as an element in $PP^{\ell}(\hat{P})$ as well, with $f(\hat{0}) = 0$ and $f(\hat{1}) = \ell$. Then for all $p \in P$, a **piecewise-linear toggle at** p is the function $\tau_p : PP^{\ell}(P) \to PP^{\ell}(P)$ defined by

$$\tau_p(f)(x) = \begin{cases} f(x) & \text{if } p \neq x, \\ \min\{f(y) : y \gtrdot p\} + \max\{f(y) : p \gtrdot y\} - f(p) & \text{if } p = x, \end{cases}$$

where we are minimizing and maximizing over $y \in \hat{P}$.

Let $p_1, \cdots, p_n$ be an linear extension of P. Then **(piecewise-linear) rowmotion** is the function $\mathrm{row}\colon PP^{\ell}(P) \to PP^{\ell}(P)$ where

$$\mathrm{row} = \tau_{p_1} \circ \cdots \circ \tau_{p_n}.$$

See Example 1.3.5 for an explicit piecewise-linear rowmotion.

In order to show that the composition does not depend on the choice of linear extension, we need the following lemma:

Lemma 1.3.2. Let $a, b \in P$. If there is no covering relation between a and b, then $\tau_a \circ \tau_b = \tau_b \circ \tau_a$.

Proof. If $a = b$, the result is trivial. So assume $a \neq b$ and there is no covering relation between a and b,

$$
\begin{aligned}
(\tau_a \circ \tau_b)(f)(x) &= \begin{cases} \tau_b(f)(x) & \text{if } x \neq a \\ \min\{\tau_b(f)(y) : y \gtrdot a\} + \max\{\tau_b(f)(y) : a \gtrdot y\} - \tau_b(f)(a) & \text{if } x = a \end{cases} \\
&= \begin{cases} \tau_b(f)(x) & \text{if } x \neq a \\ \min\{f(y) : y \gtrdot a\} + \max\{f(y) : a \gtrdot y\} - f(a) & \text{if } x = a \end{cases} \\
&= \begin{cases} f(x) & \text{if } x \neq a, b \\ \min\{f(y) : y \gtrdot b\} + \max\{f(y) : b \gtrdot y\} - f(b) & \text{if } x = b \\ \min\{f(y) : y \gtrdot a\} + \max\{f(y) : a \gtrdot y\} - f(a) & \text{if } x = a \end{cases} \\
&= (\tau_b \circ \tau_a)(f)(x)
\end{aligned}
$$

by symmetry of a and b in the penultimate line. $\square$

Proposition 1.3.3. Piecewise-linear rowmotion is independent of choice of linear extension.

Proof. Let $p_1, \cdots, p_n$ be an arbitrary linear extension of poset P and assume p_i is a minimal element. Since $p_i \leq p_j$ implies $i < j$ be definition of linear extension, we know that for all k such that $k < i$, p_k is incomparable with p_i. Then by lemma, we can swap all minimal elements of the poset P such that

$$\text{row} = \tau_{p_1} \circ \cdots \circ \tau_{p_n} = \prod_{p_i \in \min(P)} \tau_{p_i} \circ \prod_{p_i \notin \min(P)} \tau_{p_i}$$

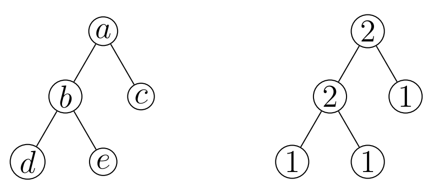
and since all minimal elements are incomparable, we know $\prod_{p_i \in \min(P)} \tau_{p_i}$ can be set in arbitrary order, while $\prod_{p_i \notin \min(P)} \tau_{p_i}$ must inherit the original order from the linear extension. Therefore, we know for a piecewise-linear rowmotion, we can always toggle all the minimal elements at the end.

Next, apply this same argument to the elements of P that are minimal in $P \setminus \min(P)$. Continue in this way, we show by induction that piecewise-linear rowmotion is well-defined independent of the choice of linear extension. $\square$

Definition 1.3.4. Let P be a poset and $f \in PP^{\ell}(P)$. An **(rowmotion) orbit** of f is the set $\{\text{row}^k(f) \colon k \in \mathbb{Z}\}$.

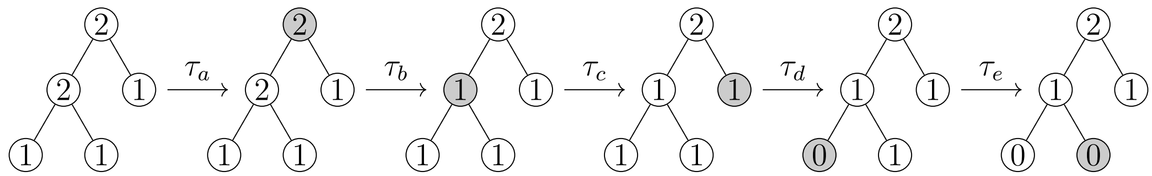
One can easily check that $PP^{\ell}(P)$ is the disjoint union of its set of orbits. We may look at some examples to see how the piecewise rowmotion is acting on the P-partitions.

Example 1.3.5. Consider the P-partition $f \in PP^2(P)$ in Example 1.1.3.

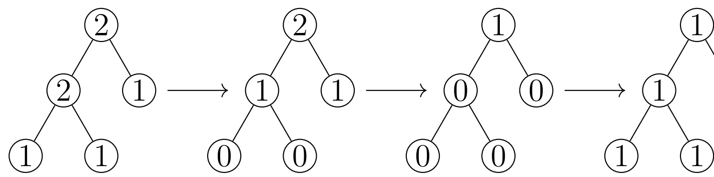


P $\qquad$ $f \in PP^2(P)$

Then (e, d, b, c, a) is a linear extension of P. and $\text{row} = \tau_e \circ \tau_d \circ \tau_b \circ \tau_c \circ \tau_a$. Compute that



and moreover, we see the following P-partitions form an orbit under the rowmotion action.



In fact, we may compute that $PP^2(P)$ has 3 orbits of length 2, 3 orbits of length 7, 1 orbit of length 4, and 1 orbit of length 22.

Example 1.3.6. When $\ell = 1$, we know $PP^1(P) \cong J(P)$ via the mapping $\alpha : I \mapsto f_I$ of Observation 1.1.7, and we may view every P-partition in $PP^1(P)$ as an order ideal. This allows us to characterize row motion on $PP^1(P)$ in terms of order ideals of P. Let I be an arbitrary order ideal and $p_1, \cdots, p_n$ be a linear extension of P, then the **rowmotion** over order ideals can be defined as

$$\begin{aligned}\text{row}^*\colon J(P) &\to J(P)\\ I &\mapsto \alpha^{-1}(\text{row}(\alpha(I)))\end{aligned}$$

so $\alpha \circ \text{row}^* = \text{row} \circ \alpha$.

Then we want to compute how this rowmotion over order ideals works explicitly. Given $f \in PP^1(P)$, let $S_f = \{x \in P : f(x) = 1 \text{ and } f(y) = 0 \text{ for all } x \gtrdot y\}$. Then

$$\text{row}(f)(x) = \begin{cases} 0 & \text{if } x \le s \text{ for some } s \in S_f, \\ 1 & \text{otherwise.} \end{cases}$$

Hence by definition of α, we know that

$$\text{row}(\alpha(I)) = \text{row}(f_I)(x) = \begin{cases} 0 & \text{if } x \le y \text{ for some } y \in \min(P \setminus I) \\ 1 & \text{otherwise} \end{cases}$$

where $\min(P \setminus I)$ is the set of minimal elements of the sub-poset $P \setminus I$. Therefore, we know the rowmotion over order ideals is

$$\begin{aligned}\text{row}^*(I) &= \alpha^{-1}(\text{row}(f_I))\\ &= \text{row}(f_I)^{-1}(\{0\})\\ &= \{x \in P : x \le y \text{ for some } y \in \min(P \setminus I)\}.\end{aligned}$$

What is exactly then the inverse of a piecewise-rowmotion?

Lemma 1.3.7. Let P be a poset and $p_1, \cdots p_n$ be a linear extension. Then

$$\text{row}^{-1} = \tau_{p_n} \circ \cdots \circ \tau_{p_1}.$$

Proof. Firstly we claim that $\tau_{p_i}^2 = \text{id}$ for all p_i. For all $x \neq p_i$, we know that $\tau_{p_i}^2(f)(x) = \tau_{p_i}(f)(x) = f(x)$, so we only need to consider the case that $x = p_i$. Then we may compute

$$\begin{aligned}\tau_{p_i}^2(f)(p_i) &= \tau_{p_i}(\tau_{p_i}(f)(p_i)) \\ &= \min\{\tau_{p_i}(f)(x) : x \gtrdot p_i\} + \max\{\tau_{p_i}(f)(x) : p_i \gtrdot x\} - \tau_{p_i}(f)(p_i) \\ &= \min\{f(x) : x \gtrdot p_i\} + \max\{f(x) : p_i \gtrdot x\} \\ &\quad - (\min\{f(x) : x \gtrdot p_i\} + \max\{f(x) : p_i \gtrdot x\} - f(p_i)) \\ &= f(p_i).\end{aligned}$$

Therefore,

$$\tau_{p_n} \circ \cdots \circ \tau_{p_1} \circ \tau_{p_1} \circ \cdots \circ \tau_{p_n} = \tau_{p_1} \circ \cdots \circ \tau_{p_n} \circ \tau_{p_n} \circ \cdots \circ \tau_{p_1} = \text{id},$$

and it follows that $\text{row}^{-1} = \tau_{p_n} \circ \cdots \circ \tau_{p_1}$. $\square$

Definition 1.3.8. Let P be a poset, and let $\mathcal{O} \subseteq PP^\ell(P)$ be an orbit under the rowmotion action. Then the **down-degree** of the orbit $\mathcal{O}$ is

$$\text{ddeg}(\mathcal{O}) = \sum_{f \in \mathcal{O}} \text{ddeg}(f).$$

We now state a result that allows us to easily calculate the sum of the down-degrees of elements in a rowmotion-orbit.

Lemma 1.3.9 (Hopkins (2020a), page 38, 39)**.** Let P be a poset and $f \in PP^\ell(P)$. Define

$$\begin{aligned}T : PP^\ell(P) &\to \mathbb{Z} \\ f &\mapsto \sum_{x \in P} (f(x) - \max\{f(a) : x \text{ covers } a \in \hat{P}\})\end{aligned}$$

Then $\text{ddeg}(f) = (T \circ \text{row}^{-1})(f)$.

Remark 1.3.10. By Lemma 1.3.9, we are provided a useful formula

$$\text{ddeg}(\mathcal{O}) = \sum_{f \in \mathcal{O}} (T \circ \text{row}^{-1})(f) = \sum_{f \in \mathcal{O}} T(f).$$

Chapter 2

Rowmotion Bijections

In this chapter, we will focus on the following conjecture raised by Sam Hopkins:

Conjecture 2.0.1. [Hopkins (2020a) Conjecture 4.38]: Let P and Q be posets such that $\mathrm{com}(P) \simeq \mathrm{com}(Q)$. Then there exists a bijection φ between the row orbits of $PP^{\ell}(P)$ and the row orbits of $PP^{\ell}(Q)$, such that for all $\mathcal{O} \subseteq PP^{\ell}(P)$, we have

1. $|\mathcal{O}| = |\varphi(\mathcal{O})|$.
2. $\mathrm{ddeg}(\mathcal{O}) = \mathrm{ddeg}(\varphi(\mathcal{O}))$.

The case $\ell = 1$ of the conjecture is known to be true (Hopkins (2020a)). In this chapter, we will (i) reformulate the proof for the case $\ell = 1$, (ii) prove the conjecture for all ℓ in the special case that Q is the dual of P, (obtained by reversing all relations in P), and (iii) conjecture a method of proof in another special case.

2.1 Rowmotion on order ideals

We start with a reformulation of the proof of the conjecture for the case $\ell = 1$. [1]

Theorem 2.1.1 (Hopkins (2020a) Proposition 4.10)**.** Let P and Q be posets such that $\mathrm{com}(P) \simeq \mathrm{com}(Q)$. Then there exists a bijection φ between the row orbits of $PP^1(P)$ and the row orbits of $PP^1(Q)$, such that for all $\mathcal{O} \subseteq PP^1(P)$, we have

1. $|\mathcal{O}| = |\varphi(\mathcal{O})|$.
2. $\mathrm{ddeg}(\mathcal{O}) = \mathrm{ddeg}(\varphi(\mathcal{O}))$.

Proof. Let row_P be the rowmotion acting on P and row_Q be the rowmotion acting on Q.

By Theorem 1.2.6, we can assume without loss of generality that P is obtained from Q by dualizing a single autonomous subset $A \subseteq P$. Also, define the following subsets

$$U = \{u \in P : u > a \text{ for all } a \in A\}$$

[1] In Hopkins (2020), the result is framed in terms of order ideals. We state it here in terms of P-partitions, as explained in Example 1.3.5.

$$
\begin{aligned}
L &= \{l \in P : l < a \text{ for all } a \in A\} \\
N &= \{n \in P : n \text{ and } a \text{ are incomparable for all } a \in A\},
\end{aligned}
$$

Let A^D be the poset obtained from A by dualizing A. Then we know $P = U \cup A \cup N \cup L$ and $Q = U \cup A^D \cup N \cup L$.

Define the complement map over the autonomous subset A as

$$
\begin{aligned}
c : PP^1(A) &\to PP^1(A^D) \\
f(x) &\mapsto 1 - f(x).
\end{aligned}
$$

For each $f \in PP^1(A)$, define the following subsets

$$
\begin{aligned}
S_f &= \{x \in A : f(x) = 1 \text{ and } f(y) = 0 \text{ for all } y \lessdot x\} \\
U_f &= \{x \in A : f(x) = 1 \text{ and } x \notin S_f\} = f^{-1}(\{1\}) \setminus S_f \\
L_f &= \{x \in A : f(x) = 0 \text{ and } x \leq y \text{ for some } y \in S_f\} \\
C_f &= \{x \in A : f(x) = 0 \text{ and } x \notin L_f\} = f^{-1}(\{0\}) \setminus L_f
\end{aligned}
$$

where we know $A = S_f \cup U_f \cup C_f \cup L_f$. Then as in Example 1.3.6,

$$
\begin{aligned}
\mathrm{row}_A(f)(x) &= \begin{cases} 0 & \text{if } x \leq s \text{ for some } s \in S_f, \\ 1 & \text{otherwise.} \end{cases} \\
&= \begin{cases} 0 & x \in S_f \cup L_f, \\ 1 & x \in U_f \cup C_f \end{cases}
\end{aligned}
$$

and hence

$$
c(\mathrm{row}_A(f))(x) = \begin{cases} 1 & x \in S_f \cup L_f, \\ 0 & x \in U_f \cup C_f \end{cases}
$$

then we can compute that

$$
\begin{aligned}
S_{c(\mathrm{row}_A(f))} &= \{x \in A^D : c(\mathrm{row}_A(f))(x) = 1 \text{ and } c(\mathrm{row}_A(f))(y) = 0 \text{ for all } x \gtrdot_{A^D} y\} \\
&= \{x \in A : c(\mathrm{row}_A(f))(x) = 1 \text{ and } c(\mathrm{row}_A(f))(y) = 0 \text{ for all } y \gtrdot_A x\}
\end{aligned}
$$

We claim that $S_{c(\mathrm{row}_A(f))} = S_f$. If $x \in S_f$, then $c(\mathrm{row}_A(f))(x) = 1$. Let $y \in A$ be an arbitrary element that covers x, then $y \in U_f$, and hence $c(\mathrm{row}_A(f))(y) = 0$. This gives us the fact that $S_f \subseteq S_{c(\mathrm{row}_A(f))}$. Conversely, if $x \in S_{c(\mathrm{row}_A(f))}$, then $c(\mathrm{row}_A(f))(x) = 1$, so $x \in S_f \cup L_f$. Assume $x \in L_f$, then there must be some $a \in S_f$ such that $a \geq x$. However, $a \in S_f$ gives us that $c(\mathrm{row}_A(f))(a) = 1$ as well, so $c(\mathrm{row}_A(f))(y) = 1$ for all $a \geq y \geq x$. This contradicts the condition that $c(\mathrm{row}_A(f))(y) = 0$ for all $y \gtrdot x$. So $x \notin L_f$, and it follows that $x \in S_f$.

Therefore, $S_{c(\mathrm{row}_A(f))} = S_f$, and

$$
\mathrm{row}_{A^D}(c(\mathrm{row}_A(f)))(x) = \begin{cases} 0 & \text{if } x \leq_{A^D} s \text{ for some } s \in S_f, \\ 1 & \text{otherwise.} \end{cases}
$$

$$= \begin{cases} 0 & \text{if } x \geq_A s \text{ for some } s \in S_f, \\ 1 & \text{otherwise.} \end{cases}$$
$$= c(f)$$

given that $f^{-1}(\{1\}) = \{x \in A : \ x \geq_A s \text{ for some } s \in S_f\}$. Therefore, we have $c \circ \mathrm{row}_A = \mathrm{row}_{A^D}^{-1} \circ c$ and the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, mapsto, "\mathrm{row}_A"] & \mathrm{row}_A^{-1}(f) \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"] & f \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"] & \mathrm{row}_A(f) \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"] & \mathrm{row}_A^2(f) \arrow[r, mapsto, "\mathrm{row}_A"] \arrow[d, mapsto, "c"] & \cdots \\
\cdots & \mathrm{row}_{A^D}(c(f)) \arrow[l, mapsto, "\mathrm{row}_{A^D}"'] & c(f) \arrow[l, mapsto, "\mathrm{row}_{A^D}"'] & \mathrm{row}_{A^D}^{-1}(c(f)) \arrow[l, mapsto, "\mathrm{row}_{A^D}"'] & \mathrm{row}_{A^D}^{-2}(c(f)) \arrow[l, mapsto, "\mathrm{row}_{A^D}"'] & \cdots \arrow[l, mapsto, "\mathrm{row}_{A^D}"']
\end{tikzcd}$$

Looking into the diagram, we discover that the upper row and the lower row form two rowmotion orbits of the same size in $PP^1(A)$ and $PP^1(A^D)$, respectively. Let $f_\varnothing$ be the P-partition mapping all poset elements to 1 and $f_A = f_{A^D}$ be the P-partition mapping all poset elements to 0. Then we can construct a bijection $\psi \colon PP^1(A) \to PP^1(A^D)$ as follows:

1. Let $\psi(f_\varnothing) = f_\varnothing$. Then, for all integer k, let $\psi(\mathrm{row}_A^k(f_\varnothing)) = \mathrm{row}_{A^D}^k(f_\varnothing)$.
2. Pick arbitrary $f \in PP^1(A)$ that has not been mapped by ψ yet. Let $\psi(f) = c(f)$ and $\psi(\mathrm{row}_A^k(f)) = \mathrm{row}_{A^D}^k(c(f))$ for all integers k.
3. Repeat step 2 until every element has been mapped.

Then we obtain a bijection satisfying that

1. For all $f \in PP^1(A)$, we have $\psi(\mathrm{row}_A(f)) = \mathrm{row}_{A^D}(\psi(f))$.
2. For all $f \in PP^1(A)$, $\psi(f)$ and $c(f)$ are in the same rowmotion orbit.
3. $\psi(f_\varnothing) = f_\varnothing$ and $\psi(f_A) = f_A$.

since c is an invertible function, and $\mathrm{row}(f_\varnothing) = f_A$.

Then we extend the bijection ψ to the following bijection:

$$\tilde{\psi} \colon PP^1(P) \to PP^1(Q)$$
$$f(x) \mapsto \begin{cases} \psi(f)(x) & \text{if } x \in A, \\ f(x) & \text{if } x \notin A. \end{cases}$$

For all $f \in PP^1(P)$, let $f(A)$ be the image of f restricted the autonomous subset A. Since A is autonomous,

1. If $f(A) = \{0, 1\}$, then $f(U) = \{1\}$ and $f(L) = \{0\}$.
2. If $f(A) = \{0\}$ or $f(A) = \{1\}$, then $\tilde{\psi}(f) = f$.

It follows that $\tilde{\psi}(f) \in PP^1(Q)$. It's a bijection since the construction of $\tilde{\psi}$ makes it an obvious injection, and $PP^1(P)$ and $PP^1(Q)$ are having the same size.

Then we claim that $\tilde{\psi} \circ \mathrm{row}_P = \mathrm{row}_Q \circ \tilde{\psi}$ holds for the extended bijection as well. Note that if $f(A) = \{0\}$ or $f(A) = \{1\}$, then $\tilde{\psi}(f) = f$ and $\tilde{\psi}(\mathrm{row}_P(f)) = \mathrm{row}_Q(\tilde{\psi}(f))$. Therefore, we only need to consider the situation that $f(A) = \{0, 1\}$. Note that

$$\begin{aligned}
\tilde{\psi}(\mathrm{row}_P(f))(x) &= \begin{cases} \psi(\mathrm{row}_P(f))(x) & \text{if } x \in A \\ \mathrm{row}_P(f)(x) & \text{if } x \notin A \end{cases} \\
&= \begin{cases} \mathrm{row}_Q(\psi(f))(x) & \text{if } x \in A \\ \mathrm{row}_Q(f)(x) & \text{if } x \notin A \end{cases} \\
&= \mathrm{row}_Q(\tilde{\psi}(f))(x),
\end{aligned}$$

and hence the bijection indeed commutes with rowmotion. Define

$$\varphi(\mathcal{O}) = \{\tilde{\psi}(f) : f \in \mathcal{O}\}$$

where $\mathcal{O}$ is an arbitrary rowmotion orbit in $PP^1(P)$. Then φ is a bijection between row-orbits and indeed $|\mathcal{O}| = |\varphi(\mathcal{O})|$.

Next we want to show $\mathrm{ddeg}(\mathcal{O}) = \mathrm{ddeg}(\varphi(\mathcal{O}))$. We know by the definition of down-degree that $\mathrm{ddeg}(f) = \mathrm{ddeg}(f^{-1}\{0\}) = |\max(f^{-1}\{0\})| = |L_f|$. Also, we know

$$\mathrm{ddeg}(\mathrm{row}_P(f)) = T(f) = \sum_{x \in P} f(x) - \max\{f(a) : x \text{ covers } a \in \hat{P}\} = |S_f|$$

Note that if we looking at the autonomous part of the rowmotion orbit $\mathcal{O}$ of any $f \in PP^1(P)$, it happens to be three cases: $f(A) = \{0\}$, $f(A) = \{1\}$ and $f(A) = \{0, 1\}$. So

$$\begin{aligned}
\mathrm{ddeg}(\mathcal{O}) &= \sum_{f \in \mathcal{O}} \mathrm{ddeg}(f) \\
&= \sum_{f \in \mathcal{O},\ f(A) = \{1\}} \mathrm{ddeg}(f) + \sum_{f \in \mathcal{O},\ f(A) = \{0\}} \mathrm{ddeg}(f) + \sum_{f \in \mathcal{O},\ f(A) = \{1,0\}} \mathrm{ddeg}(f)
\end{aligned}$$

Assume $f(A) = \{0\}$ or $f(A) = \{1\}$, then $\tilde{\psi}(f) = f$ as we defined above. Then certainly we would have $\mathrm{ddeg}(f) = \mathrm{ddeg}(\tilde{\psi}(f))$. So inside the rowmotion orbit, we only need to show that

$$\sum_{f \in \mathcal{O},\ f(A) = \{1,0\}} \mathrm{ddeg}(f) = \sum_{f \in \varphi(\mathcal{O}),\ f(A) = \{1,0\}} \mathrm{ddeg}(f)$$

If $f(A) = \{0, 1\}$ for some $f \in \mathcal{O}$, then we know $f(U) = \{1\}$ and $f(L) = \{0\}$. This forces the entire row_A orbit must be contained inside $\mathcal{O}$ when restricting to the autonomous subset A, while the corresponding orbit must be contained in $\varphi(\mathcal{O})$. Define the following function as

$$\lambda_f : P \to \{0, 1\}$$

$$p \mapsto \begin{cases} 1 & \text{if } p \in L_f \\ 0 & \text{otherwise,} \end{cases}$$

then we can also describe the down-degree of $f \in PP^1(P)$ as $\sum_{p \in P} \lambda_f(p)$. Thus when $f(A) = \{0, 1\}$, we may compute that

$$\begin{aligned} \text{ddeg}(f) &= \sum_{p \in P} \lambda_f(p) \\ &= \sum_{p \in N} \lambda_f(p) + \sum_{p \in A} \lambda_{f|_A}(p) \\ &= \sum_{p \in N} \lambda_f(p) + \text{ddeg}(f|_A) \end{aligned}$$

where the first part is equal by the construction of $\tilde{\psi}$. We know for all $f \in PP^1(A)$, $\psi(f)$ and $c(f)$ are in the same rowmotion orbit. Also we know that

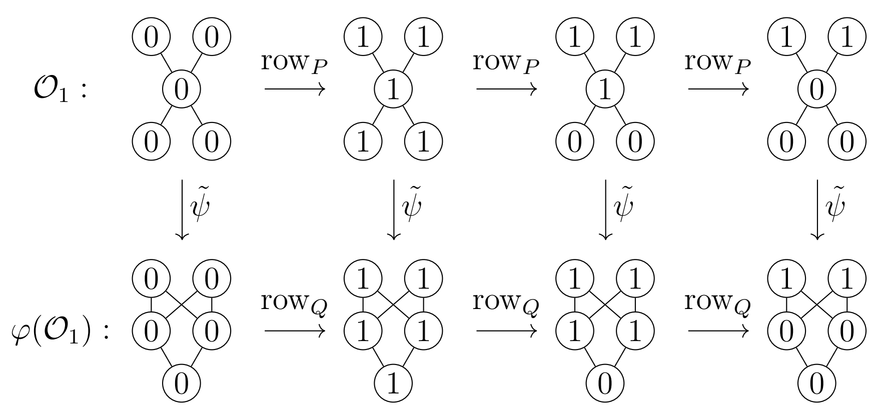
$$\text{ddeg}(\text{row}_{A^D}(c(f))) = |S_{c(f)}| = |L_f| = \text{ddeg}(f).$$

hence, we have that $\text{ddeg}(\mathcal{O}) = \text{ddeg}(\varphi(\mathcal{O}))$.

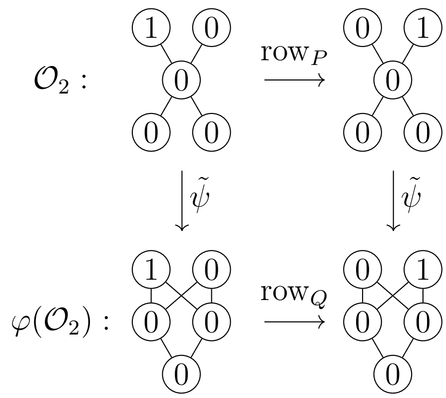
□

Example 2.1.2. Consider the posets P and Q in Example 1.2.5 where $A = \{c, d, e\}$ is their autonomous subset. We want to construct the bijection $\tilde{\psi}$.

In order to do so, we first construct the orbits of these $f \in PP^1(P)$ such that $f(A) = \{0\}$. Applying $\tilde{\psi}$, we get the orbits of $g \in PP^1(Q)$ such that $g(A^D) = \{0\}$.

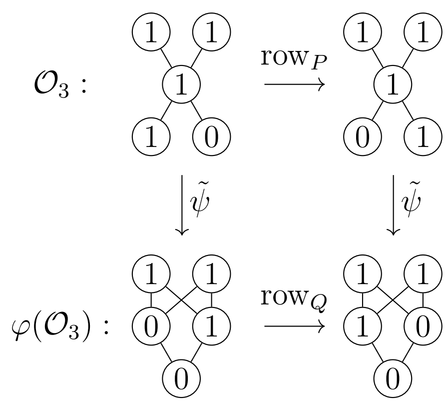


and



We may check that $\mathrm{ddeg}(\mathcal{O}_1) = \mathrm{ddeg}(\varphi(\mathcal{O}_1)) = 5$ and $\mathrm{ddeg}(\mathcal{O}_2) = \mathrm{ddeg}(\varphi(\mathcal{O}_2)) = 2$.

Then we consider an arbitrary P-partition where $f(A) = \{0, 1\}$. Then we have the following orbits matched up:



where the first correspondence is obtained by the complement map $\tilde{c}$, and the left-overs are matched up to satisfy the relation $\tilde{\psi} \circ \mathrm{row}_P = \mathrm{row}_Q \circ \tilde{\psi}$. Since the length of the orbit is 2, it happened that $\tilde{\psi} = \tilde{c}$ for all P-partitions. We can also check that $\mathrm{ddeg}(\mathcal{O}_3) = \mathrm{ddeg}(\varphi(\mathcal{O}_3)) = 2$.

2.2 Rowmotion on P-partitions

This section will discuss the general case where ℓ is an arbitrary integer. We will start with the special case that P itself is the autonomous subset.

2.2.1 When $P = A$

Theorem 2.2.1. Let A be an arbitrary poset. Let A^D be the poset obtained from A by dualizing A. Let $\ell \geq 1$. Then there exists a bijection φ between the row orbits of $PP^\ell(A)$ and the row orbits of $PP^\ell(A^D)$, such that for all $\mathcal{O} \subseteq PP^\ell(A)$, we have

1. $|\mathcal{O}| = |\varphi(\mathcal{O})|$.
2. $\displaystyle\sum_{f \in \mathcal{O}} \mathrm{ddeg}(f) = \sum_{f \in \varphi(\mathcal{O})} \mathrm{ddeg}(f)$.

Proof. Define the complement map

$$\begin{aligned} c : PP^{\ell}(A) &\to PP^{\ell}(A^D) \\ f(x) &\mapsto \ell - f(x) \end{aligned}$$

and let $p_1, \cdots, p_n$ be a linear extension of A. Then we know $p_n, \cdots, p_1$ is a linear extension of A^D.

Our first task is to show that the following diagram commutes.

$$\begin{tikzcd} PP^{\ell}(A) \arrow[r, "\text{row}"] \arrow[d, "c"] & PP^{\ell}(A) \arrow[d, "c"] \\ PP^{\ell}(A^D) & PP^{\ell}(A^D) \arrow[l, "\tilde{\text{row}}"] \end{tikzcd}$$

where $\text{row} = \tau_{p_1} \circ \cdots \circ \tau_{p_n}$ is the piecewise-rowmotion over A and $\tilde{\text{row}} = \tilde{\tau}_{p_n} \circ \cdots \circ \tilde{\tau}_{p_1}$ is the piecewise-rowmotion over A^D.

Let p be an arbitrary fixed element in A. So for any $a \in A$ such that $a \neq p$, we have

$$(c \circ \tau_p)(f)(a) = \ell - \tau_p(f)(a) = \ell - f(a) = c(f)(a) = (\tilde{\tau}_p \circ c)(f)(a),$$

and when $a = p$, we have

$$\begin{aligned} (c \circ \tau_p)(f)(p) &= \ell - \tau_p(f)(p) \\ &= \ell - (\min\{f(x) : x \gtrdot p\} + \max\{f(x) : p \gtrdot x\} - f(p)) \\ &= (\ell - \max\{f(x) : x \gtrdot p\}) + (\ell - \min\{f(x) : p \gtrdot x\}) - (\ell - f(p)) \\ &= \min\{\ell - f(x) : x \gtrdot p\} + \max\{\ell - f(x) : p \gtrdot x\} - (\ell - f(p)) \\ &= \min\{c(f)(x) : x \gtrdot p\} + \max\{c(f)(x) : p \gtrdot x\} - c(f)(p) \\ &= (\tilde{\tau}_p \circ c)(f)(p). \end{aligned}$$

Therefore, $c \circ \tau_p = \tilde{\tau}_p \circ c$. So

$$\tilde{\tau}_p \circ c \circ \tau_p = \tilde{\tau}_p^2 \circ c = c$$

since doing a toggle twice is equal to doing nothing. Then it follows that

$$\tilde{\text{row}} \circ c \circ \text{row} = \tilde{\tau}_{p_n} \circ \cdots \circ \tilde{\tau}_{p_1} \circ c \circ \tau_{p_1} \circ \cdots \circ \tau_{p_n} = c$$

telling us the diagram above indeed commutes. For all $f \in PP^{\ell}(A)$, the above commutative diagram tells us that orbit generated by $f \in PP^{\ell}(A)$ has the same length as the orbit generated by $c(f) \in PP^{\ell}(A^D)$. Therefore, as in the proof of Theorem 2.1.1, we can construct a bijection $\psi : PP^{\ell}(A) \to PP^{\ell}(A^D)$ which satisfies

1. For all $f \in PP^{\ell}(A)$, we have $\psi \circ \text{row} = \tilde{\text{row}} \circ \psi$.

2. For all $f \in PP^{\ell}(A)$, $\psi(f)$ and $c(f)$ are in the same rowmotion orbit.

In this case, we don't need to fix $\psi(f_A) = f_A$ and $\psi(f_\varnothing) = f_\varnothing$. We matched these up before in the proof of Theorem 2.1.1 in order to extend ψ to $\tilde{\psi}$. Define $\varphi(\mathcal{O}) = \{\psi(f) : f \in \mathcal{O}\}$, then indeed $|\mathcal{O}| = |\varphi(\mathcal{O})|$.

Next we need to show that the bijection preserves the down-degree. In order to do so, we look into the following commutative diagram:

$$\begin{tikzcd}
\mathrm{row}^{-1}(f) \arrow[r, mapsto, "\mathrm{row}"] \arrow[d, mapsto, "c"] & f \arrow[d, mapsto, "c"] \\
\tilde{\mathrm{row}}(c(f)) & c(f) \arrow[l, mapsto, "\tilde{\mathrm{row}}"]
\end{tikzcd}$$

By the property of linear extensions, we know that $p_i \le p_j$ implies $i < j$, so

$$\begin{aligned}
\mathrm{row}(f)(p_i) =& ((\tau_1 \circ \cdots \circ \tau_{p_{i-1}}) \circ (\tau_{p_i} \circ \cdots \circ \tau_{p_n}))(f)(p_i) \\
=& (\tau_{p_i} \circ \cdots \circ \tau_{p_n})(f)(p_i) \\
=& \min\{(\tau_{p_{i+1}} \circ \cdots \circ \tau_{p_n})(f)(p_j) : p_j \gtrdot p_i\} \\
& + \max\{(\tau_{p_{i+1}} \circ \cdots \circ \tau_{p_n})(f)(p_j) : p_i \gtrdot p_j\} - f(p_i) \\
=& \min\{(\tau_{p_j} \circ \cdots \circ \tau_{p_n})(f)(p_j) : p_j \gtrdot p_i\} + \max\{f(p_j) : p_i \gtrdot p_j\} - f(p_i) \\
=& \min\{\mathrm{row}(f)(p_j) : p_j \gtrdot p_i\} + \max\{f(p_j) : p_i \gtrdot p_j\} - f(p_i).
\end{aligned}$$

Since $p_j \gtrdot p_i$ implies that $i + 1 \le j \le n$ and $p_i \gtrdot p_j$ implies that $1 \le j < i$. The main idea is that because that when we are toggling τ_{p_i} inside of row_A, we have already toggled everything that covers p_i, while everything it covers is staying fixed. Similarly, we know

$$\mathrm{row}^{-1}(f)(p_i) = \min\{f(p_j) : p_j \gtrdot x\} + \max\{\mathrm{row}^{-1} f(p_j) : p_i \gtrdot p_j\} - f(p_i).$$

Then we may check that

$$\begin{aligned}
\mathrm{T}(\mathrm{row}^{-1}(f)) &= \sum_{p_i \in A} \mathrm{row}^{-1}(f)(p_i) - \max\{\mathrm{row}^{-1}(f)(p_j) : p_i \gtrdot_A p_j\} \\
&= \sum_{p_i \in A} \min\{f(p_j) : p_j \gtrdot_A p_i\} + \max\{\mathrm{row}_A^{-1} f(p_j) : p_i \gtrdot_A p_j\} - f(p_i) \\
&\quad - \max\{\mathrm{row}^{-1}(f)(p_j) : p_i \gtrdot_A p_j\} \\
&= \sum_{p_i \in A} \min\{f(p_j) : p_j \gtrdot_A p_i\} - f(p_i) \\
&= \sum_{p_i \in A^D} \min\{f(p_j) : p_i \gtrdot_{A^D} p_j\} - f(p_i) \\
&= \sum_{p_i \in A^D} \ell - f(p_i) - \max\{\ell - f(p_j) : p_i \gtrdot_{A^D} p_j\} \\
&= \mathrm{T}(c(f))
\end{aligned}$$

which gives us that

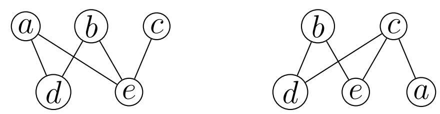
$$\mathrm{ddeg}(f) = \mathrm{T}(\mathrm{row}^{-1}(f)) = \mathrm{T}(c(f)) = \mathrm{ddeg}(\mathrm{row}(c(f))).$$

Moreover, given a correspondence of orbits as in the following diagram, the equality gives us that the down-degree of dashedly-connected P-partitions must equal

$$\begin{tikzcd}
\cdots \arrow[r, "\text{row}"] & \text{row}^{-1}(f) \arrow[r, "\text{row}"] \arrow[d, "c"] \arrow[dl, dash, dashed] & f \arrow[r, "\text{row}"] \arrow[d, "c"] \arrow[dl, dash, dashed] & \text{row}(f) \arrow[r, "\text{row}"] \arrow[d, "c"] \arrow[dl, dash, dashed] & \text{row}^2(f) \arrow[r, "\text{row}"] \arrow[d, "c"] \arrow[dl, dash, dashed] & \cdots \arrow[dl, dash, dashed] \\
\cdots & c(\text{row}^{-1}(f)) \arrow[l, "\text{row}"'] & c(f) \arrow[l, "\text{row}"'] & c(\text{row}(f)) \arrow[l, "\text{row}"'] & c(\text{row}^2(f)) \arrow[l, "\text{row}"'] & \cdots \arrow[l, "\text{row}"']
\end{tikzcd}$$

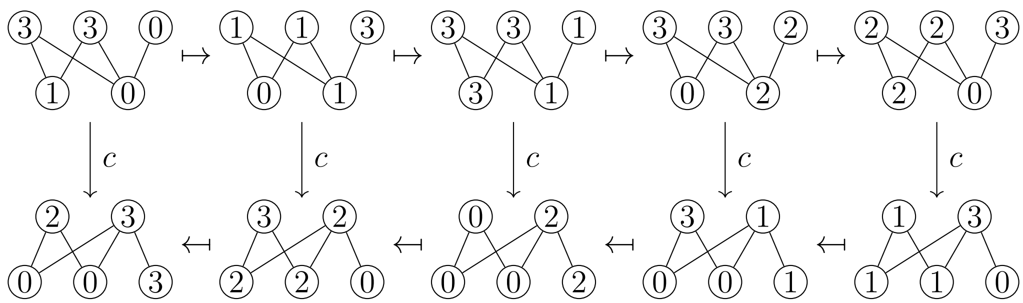
and hence we know that the down-degree of the entire orbits must equal. □

Example 2.2.2. Let P and Q be the following posets



P $\qquad$ Q

then Q is a poset obtained from P by dualizing P. Consider the following P-partitions $f \in PP^3(P)$:



We see that the P-partitions forms two orbits that are bijectively related by φ. Moreover, we can check the down-degree of each P-partitions are

$$\begin{tikzcd}
5 \arrow[r, mapsto] \arrow[d] & 5 \arrow[r, mapsto] \arrow[d] & 3 \arrow[r, mapsto] \arrow[d] & 4 \arrow[r, mapsto] \arrow[d] & 4 \arrow[d] \\
5 & 3 \arrow[l, mapsto] & 4 \arrow[l, mapsto] & 4 \arrow[l, mapsto] & 5 \arrow[l, mapsto]
\end{tikzcd}$$

where for each rectangle, the upper right corner equal to the lower left corner.

2.2.2 When $P = U \cup A \cup L$

We now assume that P has the form $U \cup A \cup L$. Thus, P is arbitrary except that $N = \varnothing$. To indicate the structure of P, we will sometimes use the notation

$$P = \begin{array}{c} U \\ | \\ A \\ | \\ L \end{array}$$

Define $\ell(f) = \min\{f(a) : a \in P \text{ covers } A\}$ and $0(f) = \max\{f(a) : A \text{ covers } a \in P\}$. Let Q be the poset obtained from P by dualizing A, and then set

$$\begin{aligned} c : PP^{\ell}(P) &\to PP^{\ell}(Q) \\ f(x) &\mapsto \begin{cases} \ell(f) + 0(f) - f(x) & \text{if } x \in A \\ f(x) & \text{if } x \notin A \end{cases} \end{aligned}$$

This construction makes sure that $c(f)$ is still a P-partition, i.e. a weakly order preserving map.

Let $U \subseteq P$ be the subposet covers A and $L \subseteq P$ be the subposet covered by A. We want that with such complement map, the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{row}^{-1}(f)|_U \arrow[d, dash, dotted] & f|_U \arrow[d, dash, dotted] & \mathrm{row}(f)|_U \arrow[d, dash, dotted] & \mathrm{row}^2(f)|_U \arrow[d, dash, dotted] & \\
\cdots \arrow[r, "\mathrm{row}_P"] & \mathrm{row}^{-1}(f)|_A \arrow[r, "\mathrm{row}_P"] \arrow[d, dash, dotted] & f|_A \arrow[r, "\mathrm{row}_P"] \arrow[d, dash, dotted] & \mathrm{row}(f)|_A \arrow[r, "\mathrm{row}_P"] \arrow[d, dash, dotted] & \mathrm{row}^2(f)|_A \arrow[r, "\mathrm{row}_P"] \arrow[d, dash, dotted] & \cdots \\
 & \mathrm{row}^{-1}(f)|_L \arrow[d, "c"] & f|_L \arrow[d, "c"] & \mathrm{row}(f)|_L \arrow[d, "c"] & \mathrm{row}^2(f)|_L \arrow[d, "c"] & \\
 & \mathrm{row}(f)|_U \arrow[d, dash, dotted] & f|_U \arrow[d, dash, dotted] & \mathrm{row}^{-1}(f)|_U \arrow[d, dash, dotted] & \mathrm{row}^{-2}(f)|_U \arrow[d, dash, dotted] & \\
\cdots & (c \circ \mathrm{row}^{-1})(f)|_A \arrow[l, "\mathrm{row}_Q"'] \arrow[d, dash, dotted] & c(f)|_A \arrow[l, "\mathrm{row}_Q"'] \arrow[d, dash, dotted] & (c \circ \mathrm{row})(f)|_A \arrow[l, "\mathrm{row}_Q"'] \arrow[d, dash, dotted] & (c \circ \mathrm{row}^2)(f)|_A \arrow[l, "\mathrm{row}_Q"'] \arrow[d, dash, dotted] & \cdots \arrow[l, "\mathrm{row}_Q"'] \\
 & \mathrm{row}(f)|_L & f|_L & \mathrm{row}^{-1}(f)|_L & \mathrm{row}^{-2}(f)|_L &
\end{tikzcd}$$

Claim: The diagram is commutative if and only if there exists an element $f \in PP^{\ell}(P)$ in each rowmotion orbit such that

$$\ell(\mathrm{row}_P^k(f)) = \ell(\mathrm{row}_P^{-k}(f)) \quad \text{and} \quad 0(\mathrm{row}_P^k(f)) = 0(\mathrm{row}_P^{-k}(f)) \tag{2.1}$$

for all integer k.

Proof. ($\Longrightarrow$) Let $p_1, \cdots, p_n$ be a linear extension of P, where $L = \{p_1 \cdots, p_i\}$, $A = \{p_i, \cdots, p_j\}$ and $U = \{p_{j+1}, \cdots, p_n\}$. Define

$$\begin{aligned} \tau_U &= \tau_{p_{j+1}} \circ \cdots \circ \tau_{p_n} \\ \tau_A &= \tau_{p_{i+1}} \circ \cdots \circ \tau_{p_j} \\ \tau_L &= \tau_{p_1} \circ \cdots \circ \tau_{p_i} \end{aligned}$$

and then $\mathrm{row}_P = \tau_L \circ \tau_A \circ \tau_U$ and $\mathrm{row}_Q = \tau_L \circ \tau_A^{-1} \circ \tau_U$. Assume x is a maximal element in A, then

$$\begin{aligned} c(\mathrm{row}_P)(f)(x) &= \ell(\mathrm{row}(f)) + 0(\mathrm{row}(f)) - (\ell(\mathrm{row}(f)) + \max\{f(y) : y \lessdot_A x\} - f(x)) \\ &= 0(\mathrm{row}(f)) - \max\{f(y) : y \lessdot_A x\} + f(x) \end{aligned}$$

and since x is a minimal element of A^D, we have

$$\begin{aligned}\mathrm{row}_Q^{-1}(c(f))(x) &= (\tau_U^{-1}\circ\tau_A\circ\tau_L^{-1})(c(f))(x)\\ &= (\tau_x\circ\tau_L^{-1})(c(f))(x)\\ &= \min\{c(f)(y) : y \gtrdot_{A^D} x\} + 0(\mathrm{row}_Q^{-1}(f)) - c(f)(x)\\ &= \min\{c(f)(y) : y \lessdot_A x\} + 0(\mathrm{row}_Q^{-1}(f)) - c(f)(x)\\ &= 0(\mathrm{row}^{-1}(f)) - \max\{f(p_i) : y \lessdot_A x\} + f(x)\end{aligned}$$

so we know that if the diagram commutes, then there has to be the condition $0(\mathrm{row}(f)) = 0(\mathrm{row}^{-1}(f))$.

A similar argument can be applied to any minimal element of A to obtain that $\ell(\mathrm{row}(f)) = \ell(\mathrm{row}^{-1}(f))$ is also a condition to let the diagram commute. Both relations can be generalized to all integers as we push through the commutative diagram.

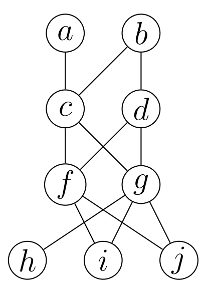
($\Longleftarrow$) It can be easily checked that if the condition holds, then $\tau_x\circ c = c\circ\tau_x$ for all $x\in A$, and hence

$$(c\circ\mathrm{row}_P^k)(f) = (\mathrm{row}_Q^{-k}\circ c)(f)$$

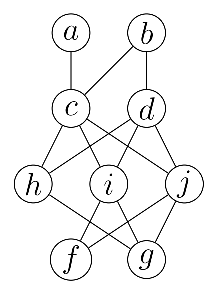
for all integer k. Hence the diagram commutes. □

Sadly, it is not true that there always exists such element, so we cannot use this complement map to extend to an bijection that commutes with the complement map.

Example 2.2.3. Let P and Q be the following posets:

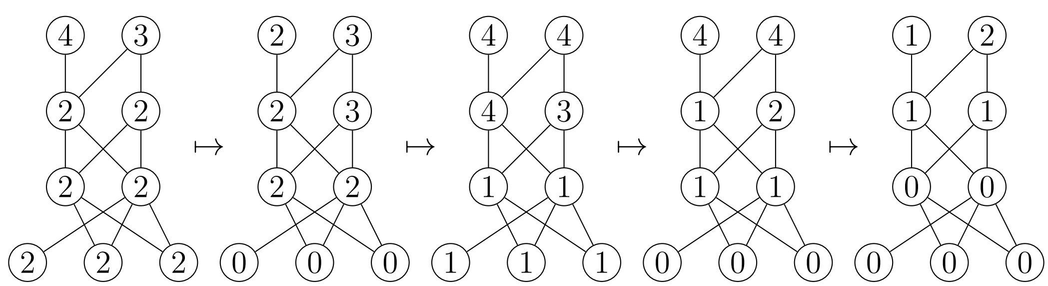


P



Q

Then Q is obtained from P by dualizing $A = \{f, g, h, i, j\}$. Consider the following P-partition orbit $\mathcal{O}\subseteq PP^4(P)$,



and we can see that $\{\ell(f) : f\in\mathcal{O}\} = \{2, 2, 3, 1, 1\}$. There's no element $f\in\mathcal{O}$ satisfying the condition in the claim.

However, while examining different P-partitions with a computer, we figured out some special pattern without any counterexamples yet.

Conjecture 2.2.4. Let $P = U \cup A \cup L$ where A is an autonomous subset and

$$U = \{u \in P : u > a \text{ for all } a \in A\}$$
$$L = \{l \in P : l < a \text{ for all } a \in A\}$$

Let Q be the poset obtained from P by dualizing A. For all $f \in PP^\ell(P)$, let $\mathcal{O}(f)$ be the rowmotion orbit generated by f, then

1. $|\mathcal{O}(f)| = |\mathcal{O}(c(f))|$.
2. $\mathrm{ddeg}(\mathcal{O}(f)) = \mathrm{ddeg}(\mathcal{O}(c(f)))$.

for all $f \in PP^\ell(P)$.

This might be true because even if the diagram is not commutative, some P-partition structure is still preserved by the rowmotion and the complement map. Hence, length of the orbit is going to be the same.

On the other hand, when computing the down-degree of an orbit, we know $\mathrm{ddeg}(\mathcal{O}) = \sum_{f \in \mathcal{O}} T(f)$. So if certain structure of the P-partition is preserved, the total down-degree is also going to be preserved.

Theorem 2.2.5 (Hall Marriage Theorem)**.** Let $G = X \cup Y$ be a finite bipartite graph. An **perfect matching** is a matching that covers all vertices of the graph with part X and Y. For $W \subseteq X$, let $N_G(W)$ be the **neighborhood** of W in G, i.e., the set of points in Y that are connected to some point in W. Then there exists a perfect matching if and only if for all subset $W \subseteq X$, $|W| \leq |N_G(W)|$.

Theorem 2.2.6. Assume Conjecture 2.2.4 holds for some $P = U \cup A \cup L$ and $Q = U \cup A^D \cup L$, then there exists a bijection φ between the row orbits of $PP^\ell(P)$ and the row orbits of $PP^\ell(Q)$, such that for all $\mathcal{O} \subseteq PP^\ell(P)$, we have

1. $|\mathcal{O}| = |\varphi(\mathcal{O})|$.
2. $\mathrm{ddeg}(\mathcal{O}) = \mathrm{ddeg}(\varphi(\mathcal{O}))$.

Proof. Assume Conjecture 2.2.4 holds. Let X and Y be the sets of rowmotion orbits of $PP^\ell(P)$ and $PP^\ell(Q)$, respectively. Let G be the graph with vertices $X \cup Y$, and with an edge between $\mathcal{O} \in X$ and $\mathcal{O}' \in Y$ if there exists $f \in \mathcal{O}$ such that $c(f) \in \mathcal{O}'$. Let $W \subseteq X$. By the conjecture, we know that $|\mathcal{O}(f)| = |\mathcal{O}(c(f))|$. So without loss of generality, we may assume that $|\mathcal{O}| = k$ for all $\mathcal{O} \in W$.

Let $K = \{f \in \mathcal{O} : \mathcal{O} \in W\} \subseteq PP^\ell(P)$, then we know $|K| = \sum_{\mathcal{O} \in W} |\mathcal{O}| = k|W|$, and $|c(K)| = |K| = k|W|$. since c is a bijection. Then

$$|N_G(W)| = |\{\mathcal{O}(f) : f \in c(K)\}| \geq \frac{c(K)}{k} = |W|$$

since orbits in $N_G(W)$ must also be of the size k. Hence by Hall Marriage Theorem, there exists a perfect matching $\varphi : PP^\ell(P) \to PP^\ell(Q)$ such that

1. $|\mathcal{O}(f)| = |\varphi(\mathcal{O}(f))|$.
2. $\mathrm{ddeg}(\mathcal{O}(f)) = \mathrm{ddeg}(\varphi(\mathcal{O}(f)))$.

$\square$

2.2.3 Decomposition of P-partitions

Here are some closing thoughts on another possible approach to proving the conjecture. The idea is to decompose each element of $PP^\ell(P)$ into ℓ elements of $PP^1(P)$. (Recall that. the conjecture has already been proved in the case where $\ell = 1$.)

Lemma 2.2.7. Let P be a poset and $f \in PP^\ell(P)$. For any $1 \le i \le \ell$, define

$$f_i : PP^\ell(P) \to \{0,1\}$$
$$f_i(x) \mapsto \begin{cases} 1 & \text{if } f(x) \ge i \\ 0 & \text{otherwise.} \end{cases}$$

then we have

1. $f = \sum_{i=1}^{\ell} f_i$.
2. $\mathrm{row}(f) = \sum_{i=1}^{\ell} \overline{\mathrm{row}}(f_i)$.
3. $\mathrm{ddeg}(f) = \sum_{i=1}^{\ell} \mathrm{ddeg}(f_i)$.

where $\overline{\mathrm{row}}$ is the rowmotion action over $PP^1(P)$.

Proof. 1. This comes immediately from the construction of f_i.

2. Let x be a maximal element in P, then we know

$$\begin{aligned}\sum_{i=1}^{\ell} \overline{\mathrm{row}}(f_i)(x) &= \sum_{i=1}^{\ell}(1 + \max\{f_i(y) : y \lessdot x\} - f_i(x)) \\ &= \ell + \sum_{i=1}^{\ell} \max\{f_i(y) : y \lessdot x\} - \sum_{i=1}^{\ell} f_i(x) \\ &= \ell + \max\{f(y) : y \lessdot x\} - f(x) \\ &= \mathrm{row}(f)(x).\end{aligned}$$

   Thus by induction, we know $\mathrm{row}(f) = \sum_{i=1}^{\ell} \overline{\mathrm{row}}(f_i)$.

3. Note that $f_i \in PP^1(P)$ for all i, thus we may obtain the corresponding order ideal through the mapping α of Observation 1.1.7. By definition of down-degree, we know that

$$\mathrm{ddeg}(f) = \sum_{i=0}^{\ell-1} \mathrm{ddeg}(f^{-1}\{0, \cdots, i\})$$

$$= \sum_{i=0}^{\ell-1} \mathrm{ddeg}(\alpha^{-1}(f_{i+1}))$$
$$= \sum_{i=1}^{\ell} \mathrm{ddeg}(\alpha^{-1}(f_i))$$
$$= \sum_{i=1}^{\ell} \mathrm{ddeg}(f_i)$$

□

Conjecture 2.2.8. Let P be a ranked poset, then $\mathrm{row}(f_i) \in \{\mathrm{row}(f)_i : 1 \leq i \leq \ell\}$.

Observation 2.2.9. If $0 \in \mathrm{row}(f)(P)$, then $\ell \in f(P)$.

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