Math 411

Goal: Description of the cohomology ring for $G_r\mathbb{P}^n$.

Background An algebraic set is the set of solutions to a system of polynomial equations. An algebraic set is irreducible if it cannot be written as a proper union of algebraic sets. An irreducible algebraic set is a variety.

Example: $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$ is a reducible algebraic set.

There is a reasonable way to define the dimension of a variety.

Let X be a variety. An r-cycle on X is a finite formal sum,

$\sum n_i V_i$ where $n_i \in \mathbb{Z}$ and V_i is an r-dimensional subvariety of X.

Notation: $Z_r(X) =$ all r-cycles on X.

A subvariety $V \subseteq X$ has codimension r if $\dim Y = \dim X - r$.

There is a notion of rational equivalence among subvarieties of X which is an algebraic version of homotopy equivalence. If subvarieties V, W are rationally equivalent, we write $V \sim W$.

Rational equivalence preserves dimension.

Def. Suppose X is nonsingular. smooth Let $A^r X := Z_{n-r}(X)/\sim$.

The **Chow ring** of X is $A^* X = \bigoplus A^r X$.

To define the ring structure on $A^* X$ let $[V] \in A^r X$, $[W] \in A^s X$.

Then let $[V] \cdot [W] = [V \cap W] \in A^{r+s} X$ where the intersection is done after deforming V and W so that they meet transversally everywhere.

(V and W meet transversally at $p \in V \cap W$ if the tangent spaces of V and W at p together span the tangent space of X.)

Example $X = \mathbb{P}^n$

Take a generic linear subspace $L \subseteq \mathbb{P}^n$ such that $\dim L + \dim V = n$. Then

$$A^*\mathbb{P}^n \cong \mathbb{Z}[t]/(t^{n+1})$$

$$V \longmapsto (\deg V)\, t^{\operatorname{codim} V}$$

$\deg V = \# L \cap V$ (number of points of intersection)

$n = 2$

zero set

degree $=2$

degree $=3$

Let $p, q \in \mathbb{P}^2$, $X = Z(yz - x^2)$, $Y = (zy^2 - x^3 - zx^2)$. Then

- $2[p] + 3[q] - [X] + 4[Y] + 7[\mathbb{P}^2] \longmapsto 2t^2 + 3t^2 - 2t + 4(3t) + 7$
  $= 5t^2 + 10t + 7$
- $[X][Y] \to (2t)(3t) = 6t^2$ (X and Y meet in 6 points, at least after a continuous deformation)
- $(2[p] + [X] + [\mathbb{P}^2])^2 \to (2t^2 + t + 1)^2 = 5t^2 + 2t + 1$.

Grassmannians

Def. A sequence $A_0 \subsetneq \cdots \subsetneq A_r$ of linear subspaces of $\mathbb{P}^n$ is called a flag.

Def. Fixing a flag in $\mathbb{P}^n$ as above. The corresponding Schubert variety is

$$\Sigma(A_0, \ldots, A_r) = \{ L \in G_r \mathbb{P}^n : \dim(L \cap A_i) \geq i \ \ \forall i \}.$$

Prop. Consider $G_r \mathbb{P}^n \subseteq \mathbb{P}^N$, $N = \binom{n}{r} - 1$, via the Plücker embedding. Then

$$\Sigma(A_0, \ldots, A_r) = G_r \mathbb{P}^n \cap M$$

for some linear subspace M of $\mathbb{P}^n$. The space M is a hyperplane iff $\dim A_0 = n-r-1$ and $\dim A_i = n-r+i$ for $i = 1, \ldots, r$.

So $(\dim A_0, \dim A_1, \ldots, \dim A_{r-1}, \dim A_r) = (n-r-1, n-r+1, \ldots, n-1, n)$.

← one less than expected from the rest of the sequence.

Prop. If $A_0 \subsetneq \cdots \subsetneq A_r$ and $B_0 \subsetneq \cdots \subsetneq B_r$ are flags in $\mathbb{P}^n$ with $\dim A_i = \dim B_i$ $\forall i$, then

$$[G(A_0, \ldots A_r)] = [G(B_0, \ldots, B_r)] \in A^* G_r \mathbb{P}^n.$$

Notation If $a_i = \dim A_i$ $\forall i$, we write $G(a_0, \ldots, a_r)$ or just $(a_0, \ldots, a_r)$ for the cycle class $[G(A_0, \ldots, A_r)] \in A^* X$.

☆ **Thm.** $A^* G_r \mathbb{P}^n$ is a free abelian group on

$$\{(a_0, \ldots, a_r) : 0 \leq a_0 < \cdots < a_r \leq n\},$$

and $(a_0, \ldots, a_r) \in A^\ell X$ where $\ell = (r+1)(n-r) - \sum_{i=0}^r (a_i - i)$.

☆ Hilbert problem.

Example $G_2 \mathbb{P}^5$ (planes in 5-space)

$a_0 = 1,\ a_1 = 3,\ a_2 = 4$:

$$G(A_0, A_1, A_2) = \{L \in G_2\mathbb{P}^5 : \dim(L \cap A_i) \geq i \quad \forall i\}.$$

$\dim L \cap A_0 \geq 0 \Rightarrow L$ meets the line A_0 in a point

$\dim L \cap A_1 \geq 1 \Rightarrow L$ meets the solid A_1 in a line.

$\dim L \cap A_2 \geq 2 \Rightarrow L$ meets the 4-plane A_2 in a space of dimension 2.

Since $\dim L = 2$, this means L lies in A_2.

Prop. Let L, M be linear subspaces of $\mathbb{P}^n$. Then

$$(\ast) \qquad \dim L \cap M \geq \dim L + \dim M - n$$

Pf/ HW.

Example $a_0 = 3$, $a_1 = 4$, $a_2 = 5$

$\dim L \cap A_0 \geq 0$ is the condition that the plane L meet a the solid A_0 at least a point. By the Proposition, this is automatic since

$$\dim L \cap A_0 \geq \dim L + \dim A_0 - n$$
$$= 2 + 3 - 5 = 0.$$

Similarly $\dim L \cap A_1 \geq 1$ and $\dim A_2 \geq 2$ are automatic in this case. So $G(A_0, A_1, A_2) = (3,4,5) = [G_2 \mathbb{P}^5]$ (no condition imposed).

Example $a_0 = 1,\ a_1 = 4,\ a_2 = 5$

$(1,4,5)$ represents the condition that a 2-plane meets a given line in a point. The conditions $\dim(L \cap A_1) \geq 1$ and $\dim(L \cap A_2) \geq 2$ are non-conditions by $(\star)$ again.