Math 411

Let V be a finite-dimensional vector space over $\mathbb{R}$.

Def. A bilinear form $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$ is **nondegenerate** if

$$\langle v, w \rangle = 0 \ \ \forall w \in V \Rightarrow v = 0 \quad \text{and} \quad \langle v, w \rangle = 0 \ \ \forall v \in V \Rightarrow w = 0.$$

Prop. If $\langle\, ,\, \rangle$ is nondegenerate, we get an isomorphism

$$\begin{aligned} V &\xrightarrow{\ b\ } V^* \\ v &\longmapsto \langle v, \cdot \rangle . \end{aligned}$$

Conversely, given an isomorphism $f : V \xrightarrow{\cong} V^*$, we get a nondegenerate bilinear form

$$\begin{aligned} \langle\, ,\, \rangle : V \times V &\to \mathbb{R} \\ \langle v, w \rangle &\mapsto f(v)(w) \end{aligned}$$

Pf/ Exercise. $\square$

Given a bilinear form $\langle\, ,\, \rangle$ on V and a basis $v_1, \ldots, v_n$ for V, define the $n \times n$ matrix

$$G := (\langle v_i, v_j \rangle)$$

Prop. (1) If $u = \sum a_i v_i$ and $v = \sum b_i v_i$, then $\langle u, v \rangle = [a_1 \cdots a_n]\, G \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

(2) $\langle\, ,\, \rangle$ is nondegenerate iff $\det G \neq 0$.

(3) $\langle\, ,\, \rangle$ is symmetric iff G is symmetric.

Def. A scalar product on V is a symmetric, nondegenerate bilinear form.

Thm. (Sylvester inertia theorem) Let $\langle\, ,\, \rangle$ be a scalar product on V. Then there exists a basis for V such that the matrix for V with respect to this basis is diagonal with the form:

$$G = \operatorname{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{s})$$

The integer s is called the index of $\langle \, , \, \rangle$ and is independent of the choice of basis.

Def. A basis $v_1, \ldots, v_n$ for V with bilinear form $\langle \, , \, \rangle$ is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \pm 1 & \text{if } i = j \end{cases}.$$

We'll write $\langle v_i, v_i \rangle = \varepsilon_i \in \{-1, 1\}$ in this case.

Note: V has an orthonormal basis with respect to $\langle \, , \, \rangle$ iff $\langle \, , \, \rangle$ is a scalar product. (Think Gram-Schmidt.)

Tensors

Let $\langle , \rangle$ be a symmetric bilinear form on V. By the universal property of the symmetric algebra,

$$v, w \mapsto v \cdot w$$

$$\begin{tikzcd}
 & S^2V \arrow[d, dashed, "\exists!\, s \,\in\, (S^2V)^* \,\approx\, S^2V^*"] \\
V \times V \arrow[ur] \arrow[r, "\langle , \rangle"'] \arrow[ur, phantom, shift right=3ex, "\star"] & \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2V)^* \\
(S^kV)^* \arrow[u] & S^kV^* \arrow[l, "\star"']
\end{tikzcd}$$

$$\left[v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\pi \in S_k} \alpha_{\pi(1)}(v_1) \cdots \alpha_{\pi(k)}(v_k) \right] \longleftarrow\!\!\!\shortmid\ \alpha_1 \cdots \alpha_k$$

Example $V = T_pM$ with basis $e_1 = \frac{\partial}{\partial x}\Big|_p$, $e_2 = \frac{\partial}{\partial y}\Big|_p$, $e_3 = \frac{\partial}{\partial z}\Big|_p$, $e_4 = \frac{\partial}{\partial t}\Big|_p$

and dual basis $e_1^* = dx_p$, $e_2^* = dy_p$, $e_3^* = dz_p$, $e_4^* = dt_p$ for T_p^*M.

Then $dx^2 + dy^2 + dz^2 - dt^2 \in S^2 T_p^* M$ corresponds with a scalar product on T_pM. We have

$= (e_1^*)^2 + (e_2^*)^2 + (e_3^*)^2 - (e_4^*)^2$

$$T_pM \times T_pM \longrightarrow S^2 T_pM \longrightarrow$$

$$(e_i, e_j) \longmapsto e_ie_j \longrightarrow \left[(e_1^*)^2 + (e_2^*)^2 + (e_3^*)^2 - (e_4^*)^2\right](e_ie_j)$$

$e_1^* = \alpha_1$ $\quad$ $e_1^* = \alpha_2$ $\quad$ $\psi_1 = e_2^*$, $\psi_2 = e_2^*$

$$= \frac{1}{2}\sum_{\pi \in S_2 = \{(),(12)\}} \alpha_{\pi(1)}(e_i)\,\alpha_{\pi(2)}(e_j) + \frac{1}{2}\sum_{\pi \in S_2} \psi_{\pi(1)}(e_i)\,\psi_{\pi(2)}(e_j) + \text{etc.}$$

$$= e_1^*(e_i)\,e_1^*(e_j) + e_2^*(e_i)\,e_2^*(e_j) + e_3^*(e_i)\,e_3^*(e_j) - e_4^*(e_i)\,e_4^*(e_j)$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \neq 4 \\ -1 & \text{if } i = j = 4 \end{cases}$$

Corresponding matrix: $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = G.$

corresponds with

Prop. Suppose $\langle , \rangle$ is a scalar product on V. Then

$$I : \Lambda^k V \xrightarrow{\flat} \Lambda^k V^* \xrightarrow{\cong} (\Lambda^k V)^*$$

gives a scalar product

$$\langle , \rangle : \Lambda^k V \times \Lambda^k V \longrightarrow \mathbb{R}$$
$$(\omega, \eta) \longrightarrow I(\omega)(\eta)$$

Of course, the isomorphism $\flat$ then induces a scalar product on $\Lambda^k V^*$, too.

Pf/

Bilinearity: straightforward.

Symmetry: $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = (v_1^\flat \wedge \cdots \wedge v_k^\flat)(w_1 \wedge \cdots \wedge w_k)$

$$= \det(v_i^\flat(w_j)) \overset{\text{def. of } \flat}{=} \det(\langle v_i, w_j \rangle) = \det(\langle w_j, v_i \rangle)$$

$$= (w_1^\flat \wedge \cdots \wedge w_k^\flat)(v_1 \wedge \cdots \wedge v_k) = \langle w_1 \wedge \cdots \wedge w_k, v_1 \wedge \cdots \wedge v_k \rangle$$

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Nondegeneracy: Let $e_1, \ldots, e_n$ be an orthonormal basis for V.

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \varepsilon_i \in \{-1, 1\} & \text{if } i = j \end{cases} = \varepsilon_i \delta_{ij}$$

Then $\{e_\mu\}_{\mu_1 < \cdots < \mu_k}$ is an orthonormal basis for $\Lambda^k V$:

$$\langle e_\mu, e_\nu \rangle = \det(\langle e_{\mu_i}, e_{\nu_j} \rangle) = \begin{cases} 0 & \text{if } \mu \neq \nu \\ \prod_{i=1}^{k} \varepsilon_{\mu_i} \in \{-1, 1\} & \text{if } \mu = \nu \end{cases}.$$

Since $\Lambda^k V$ has an orthonormal basis with respect to $\langle , \rangle$, we have that $\langle , \rangle$ is a scalar product. $\square$