①

Math 411

$\iota_u : U \to M$, $\iota_v : V \to M$

$j_u : U \cap V \to U$, $j_v : U \cap V \to V$

Continuing from last time:

$$(\star)\quad 0 \to \Omega^k M \longrightarrow \Omega^k U \times \Omega^k V \longrightarrow \Omega^k(U \cap V) \to 0$$

$$\omega \longmapsto (\iota_u^* \omega, \iota_v^* \omega)$$

$$(\xi, \eta) \longmapsto j_u^* \xi - j_v^* \eta$$

Thm. $(\star)$ is a short exact sequence.

Pf/ At $\Omega^k M$: If the restriction of the form $\omega \in \Omega^k M$ to U and to V is zero, then $\omega = 0$. This is obvious by taking local coordinates. Hence, $\Omega^k M \to \Omega^k U \times \Omega^k V$ is injective.

At $\Omega^k U \times \Omega^k V$: Let $(\xi, \eta) \in \Omega^k U \times \Omega^k V$ and suppose $j_u^* \xi - j_v^* \eta = 0$.

Define $\omega \in \Omega^k M$ by $\omega(p) = \begin{cases} \xi(p) & \text{if } p \in U \\ \eta(p) & \text{if } p \in V \end{cases}$. Then $\omega \mapsto (\xi, \eta)$

under restriction. Thus, $\ker(\Omega^k U \times \Omega^k V \to \Omega^k(U \cap V)) \subseteq \operatorname{im}(\Omega^k M \to \Omega^k U \times \Omega^k V)$.

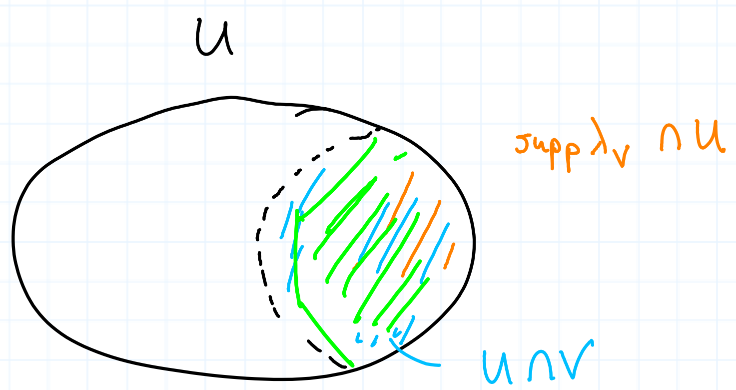
For the opposite inclusion, given any $\omega \in \Omega^k M$, we have

$\omega \mapsto (\iota_U^* \omega, \iota_V^* \omega) \mapsto j_U^* \iota_U^* \omega - j_V^* \iota_V^* \omega = (\iota_U \circ j_U)^* \omega - (\iota_V \circ j_V)^* \omega = 0$ since

$\iota_U \circ j_U(x) = \iota_V \circ j_V(x) \quad \forall x \in U \cap V$.

At $\Omega^k U \cap V$ Let $\omega \in \Omega^k(U \cap V)$. Take a partition of unity subordinate to the covering $\{U, V\}$ of M. That is take functions, $\lambda_U, \lambda_V : M \to \mathbb{R}$ with $\operatorname{supp} \lambda_U \subset U$, $\operatorname{supp} \lambda_V \subset V$ and $\lambda_U(m) + \lambda_V(m) = 1$ for all $m \in M$. Define $\omega_U = \lambda_V \omega \in \Omega^k U$ and $\omega_V = \lambda_U \omega \in \Omega^k V$. Thus, for instance, $\omega_U(p) = 0$ for $p \in U \setminus V$.

Then $(\omega_U, -\omega_V) \mapsto \omega_U - (-\omega_V) = \omega_U + \omega_V = \lambda_U \omega + \lambda_V \omega = (\lambda_U + \lambda_V)\omega = \omega$.



$\square$

The short exact sequence of chain complexes in the theorem induces a long exact sequence of cohomology groups:

Vector Fields on Spheres

Def. A vector field on a manifold M is a smooth section of the tangent bundle:

$$v : M \longrightarrow TM,$$
$$p \longmapsto v_p \in T_pM$$

$\square$

Locally, $v(p) = \sum a_i(p) \frac{\partial}{\partial x_i}\Big|_p$ where a_i is a smooth $\mathbb{R}$-valued function.

Thm. (Hairy ball thm.) Every vector field on an even-dimensional sphere has at least one zero.

Pf/ Suppose v is a non-vanishing v.f. on $S^n = \{x \in \mathbb{R}^n : \sum x_i^2 = 1\}$, n even. Then the antipodal mapping $\tau : S^n \to S^n$, $\tau(p) = -p$, is homotopic to the identity mapping:

$$h(t,x) = x\cos(t\pi) + \frac{v(x)}{|v(x)|}\sin(t\pi), \qquad t \in [0,1].$$

$h(0,x) = x$
$h(1,x) = -x$

Here we are identifying $T_x S^n$ with vectors perpendicular to x in $\mathbb{R}^{n+1}$ (thinking of tangent vectors in terms of curves through x). Thus, for fixed x, the image of $h(t,x)$ is a circle in the plane spanned by the perpendicular vectors x and $v(x)$.

By the homotopy invariance theorem, if $\omega \in \Omega^n S^n$ (automatically a cocycle since $\dim S^n = n$), then $\tau^* = id^* : H^n S^n \to H^n S \implies$ $\tau^* \omega = id^* \omega = \omega \mod d(\Omega^{n-1} S^n)$, i.e. $\tau^* \omega - \omega = d\eta$ for some $\eta \in \Omega^{n-1} S^n$. Therefore,

$$\int_{S^n} \tau^* \omega - \int_{S^n} \omega = \int_{S^n} d\eta = \int_{\partial S^n} \eta = \int_{\emptyset} \eta = 0,$$

i.e.,

$$\int_{S^n} \tau^* \omega = \int_{S^n} \omega.$$

See below for further explanation

However, if n is even, τ^* is an orientation reversing diffeomorphism☆, so

$$\int_{S^n} \tau^* \omega = -\int_{S^n} \omega.$$

So on an even-dimensional sphere, we have shown that $\int_{S^n} \omega = 0$

for all $\omega \in \Omega^n S^n$. However, this is not true — consider a bump function. Therefore, v cannot exist.

☆ 1. τ^* is orientation reversing. Let D^{n+1} denote the $(n+1)$-dimensional unit ball in $\mathbb{R}^{n+1}$. The orientation on D^{n+1} is induced by the positive orientation on $\mathbb{R}^{n+1}$. The orientation on $S^n = \partial D^{n+1}$ is the orientation induced boundary orientation: $v_1, \dots, v_n \in T_p S^n$ is positive if $v, v_1, \dots, v_n$ is positive for any outward-pointing tangent vector v. The antipodal map $\tau(x_1, \dots, x_{n+1}) = (-x_1, \dots, -x_{n+1})$ has Jacobian with determinant -1 if n is even, thus, is orientation reversing on $\mathbb{R}^{n+1}$. Therefore, it is orientation reversing on D^{n+1} and $\partial D^{n+1} = S^n$.

$$\boxed{\begin{tikzcd} S^n \arrow[d, "\tau"] & \subseteq & D^{n+1} \arrow[d, "\tau"] & \subseteq & \mathbb{R}^{n+1} \arrow[d, "\tau"] \\ S^n & \subseteq & D^{n+1} & \subseteq & \mathbb{R}^{n+1} \end{tikzcd}}$$

2. If $f: M \to N$ is an orientation-reversing diffeomorphism between n-manifolds and $\omega \in \Omega^n N$, then $\int_M f^*\omega = -\int_N \omega$.

This is a local question, so we may assume that M and N open subsets of $\mathbb{R}^n$ with the usual positive orientation and $\omega = a \, dx_1 \wedge \cdots \wedge dx_n$. Then

$$\int_M f^*\omega = \int_M a\circ f \; df_1 \wedge \cdots \wedge df_n = \int_M a\circ f \; \det(Jf) \, dx_1 \wedge \cdots \wedge dx_n$$

$$= \int_M a\circ f \; \det(Jf) = -\int_M (a\circ f) \, |\det Jf|$$

$\det Jf < 0$ since f is orientation-reversing

change of variables formula

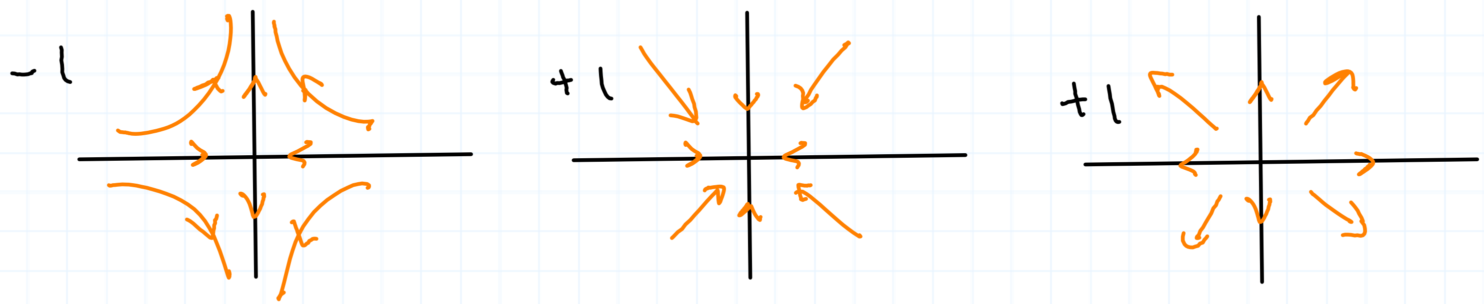
$$= -\int_N a = -\int_N \omega. \quad \square$$

What about the number of zeros of a vector field?

Let M be a compact Riemann surface (a multi-holed donut).

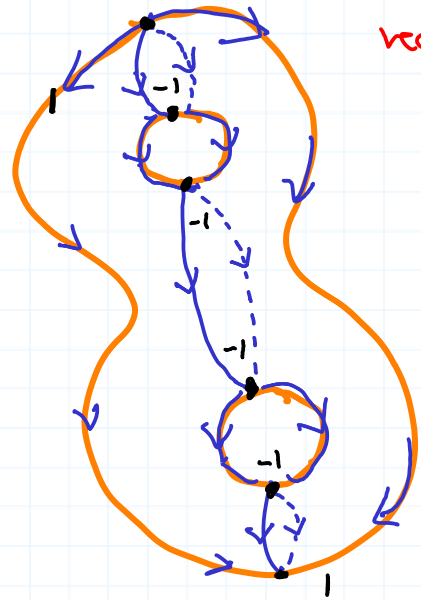
Quasi-Def. Indices of zeroes of vector fields on a surface = (change in angle the vector field makes as one travels counter-clockwise along a simple closed curve containing the zero and only that zero) / 2π.

Example



Thm. Consider a vector field on M with a finite number of zeros. Then the sum of the indices of a vector field on M is $2-2g$ where g is the genus (number of holes) of M.

Pouring-glaze-on-a-donut vector field



Note: $H^1 M \simeq \mathbb{R}^{2g}$ (use Mayer-Victoris)