Math 411 de Rham cohomology

Functors

* manifolds $\longrightarrow$ graded, anti-commutative, differential, $\mathbb{R}$-algebras

$$M \longmapsto \Omega^\bullet M := \bigoplus_{k\geq 0} \Omega^k M$$

* manifolds $\longrightarrow$ chain complexes

$M \longmapsto$ de Rham complex for M

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \cdots$$

functions $M \to \mathbb{R}$

There are contravariant functors, e.g. $f: M \to N$ induces

$$\begin{array}{ccccccccc} 0 & \to & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N & \xrightarrow{d} & \cdots \\ & & \downarrow f^* & /// & \downarrow f^* & /// & \downarrow f^* & /// & \\ 0 & \to & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M & \xrightarrow{d} & \cdots \end{array}$$

de Rham cohomology

k^{th} cohomology group: $\quad H^k M = \dfrac{\ker\left(\Omega^k M \xrightarrow{d} \Omega^{k+1} M\right)}{\operatorname{im}\left(\Omega^{k-1} M \xrightarrow{d} \Omega^k M\right)}$

closed forms = cocycles

$\leftarrow$ exact forms = coboundaries

Note: The cohomology groups measure the exactness of the de Rham complex.

If $\omega \in \Omega^k M$ is a cocycle, i.e., $d\omega = 0$, the cohomology class of ω is $[\omega] = \omega + d(\Omega^{k-1} M)$.

Cocycles $\omega, \eta \in \Omega^k M$ are cohomologous if $[\omega] = [\eta]$, i.e., if $\omega - \eta = d\alpha$ for some $\alpha \in \Omega^{k-1} M$.

Example $\quad M = \mathbb{R}$

ker = constant functions $\qquad$ ker $= \Omega^1 \mathbb{R}$

$$0 \to \Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R} \xrightarrow{d} 0$$

$\Omega^0 \mathbb{R}$ = functions $\mathbb{R} \to \mathbb{R}$ $\qquad \ker\left(\Omega^0 \mathbb{R} \to \Omega^1 \mathbb{R}\right)$ = constant functions $\cong \mathbb{R}$

$\Omega^1 \mathbb{R} = \{ \underline{f(x)}\, dx \}$

any function $\mathbb{R} \to \mathbb{R}$

(3)

$$H^0\mathbb{R} = \frac{\ker(\Omega^0\mathbb{R} \xrightarrow{d} \Omega^1\mathbb{R})}{\operatorname{im}(0 \to \Omega^0\mathbb{R})} = \frac{\text{constant functions}}{(0)} \cong \mathbb{R}$$

$$H^1\mathbb{R} = \frac{\ker(\Omega^1\mathbb{R} \to 0)}{\operatorname{im}(\Omega^0\mathbb{R} \to \Omega^1\mathbb{R})} = \frac{\Omega^1\mathbb{R}}{\Omega^1\mathbb{R}} = 0$$

What is $\operatorname{im}(\Omega^0\mathbb{R} \to \Omega^1\mathbb{R})$? Given $f(x)dx \in \Omega^1\mathbb{R}$, define $g(x) = \int_0^x f(t)dt$.

Then $g \in \Omega^0\mathbb{R}$ and $dg = f(x)dx$. Thus, $\operatorname{im}(\Omega^0\mathbb{R} \to \Omega^1\mathbb{R}) = \Omega^1\mathbb{R}$

Example $M = \mathbb{R}^3$

$$0 \to \Omega^0\mathbb{R}^3 \to \Omega^1\mathbb{R}^3 \longrightarrow \Omega^2\mathbb{R}^3 \longrightarrow \Omega^3\mathbb{R}^3 \to 0$$

$$f \mapsto \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$f_1dx + f_2dy + f_3dz \mapsto \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\wedge dy + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)dx\wedge dz + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)dx\wedge dz$$

$$f_1\,dx\wedge dy - f_2\,dx\wedge dz + f_3\,dy\wedge dz \mapsto \left(\frac{\partial f_1}{\partial z} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)dx\wedge dy\wedge dz$$

Notice the eerie relation of these maps to grad, curl, and div, respectively.

$$H^0\mathbb{R}^3 = \frac{\ker(\Omega^0\mathbb{R}^3 \to \Omega^1\mathbb{R}^3)}{\operatorname{im}(0 \to \Omega^0\mathbb{R}^3)} = \ker(\Omega^0\mathbb{R}^3 \to \Omega^1\mathbb{R}^3)$$

Let $f \in \Omega^0\mathbb{R}$. Then $df = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_3}{\partial z}dz$, and

$\operatorname{grad} f = \nabla f = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}\right)$. If $\nabla f = 0$, then $f =$ constant.

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

(Proof: For any $p \in \mathbb{R}^3$, define $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(t) = f(tp)$. By the chain rule $\gamma'(t) = \nabla f(tp) \cdot p = 0 \Rightarrow \gamma$ constant $\Rightarrow f(0) = \gamma(0) = \gamma(1) = f(p)$. Hence, $f(p) = f(0)$ $\forall p \in \mathbb{R}^3$.)

Thus, $\ker(\Omega^0\mathbb{R}^3 \to \Omega^1\mathbb{R}^3) \cong \mathbb{R}$ and $H^0\mathbb{R}^3 = \mathbb{R}$.

$$H^1\mathbb{R}^3 = \frac{\ker(\Omega^1\mathbb{R}^3 \to \Omega^2\mathbb{R}^3)}{\operatorname{im}(\Omega^0\mathbb{R}^3 \to \Omega^1\mathbb{R}^3)}$$

It turns out that every vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\operatorname{curl} F = \nabla \times F = \vec{0}$ has a potential, i.e., a function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla\varphi = F$.

To define $\varphi(p)$, pick any curve $\alpha : [0,1] \to \mathbb{R}^3$ with $\alpha(0) = \vec{0}$ and $\alpha(1) = p$. Then define $\varphi(p) = \int_\alpha F := \int_0^1 F(\alpha(t)) \cdot \alpha'(t)\, dt$ = flow of F along α.

If curl $F = 0$, then the definition of $\varphi(p)$ is independent of the particular γ chosen (by Stokes', almost).

The upshot is that $H^1 \mathbb{R}^3 = 0$. It turns out that

Thm.

$$H^k \mathbb{R}^n = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Pf/ Hmm... To come $\square$

Goal: Compute $H^k M$'s.

Thm. $H^\bullet M := \bigoplus_{k \geq 0} H^k M$ is a graded, anti-commutative algebra via the wedge product. In other words,

$$\begin{aligned} H^r M \times H^s M &\longrightarrow H^{r+s} M \\ ([\omega], [\eta]) &\longmapsto [\omega \wedge \eta] \end{aligned}$$

is well-defined.

Pf/ First note that if ω and η are cocycles, then so is $\omega \wedge \eta$:
$d(\omega \wedge \eta) = d\omega \wedge \eta \pm \omega \wedge d\eta = 0$. Next, let $\tilde{\omega} \in \Omega^{r-1} M$ and $\tilde{\eta} \in \Omega^{s-1} M$. Then

$$(\omega + d\tilde{\omega}) \wedge (\eta + d\tilde{\eta}) = \omega \wedge \eta + \omega \wedge d\tilde{\eta} + d\tilde{\omega} \wedge \eta + d\tilde{\omega} \wedge d\tilde{\eta}.$$

We need to show $\omega \wedge d\tilde{\eta}$, $d\tilde{\omega} \wedge \eta$, and $d\tilde{\omega} \wedge d\tilde{\eta}$ are coboundaries.

By the product rule, $d(\omega \wedge \tilde{\eta}) = d\omega \wedge \tilde{\eta} \pm \omega \wedge d\tilde{\eta} = \pm \omega \wedge d\tilde{\eta}$ (7)
since ω is a cocycle. Hence, $\omega \wedge d\tilde{\eta}$ and, similarly, $d\tilde{\omega} \wedge \eta$ are coboundaries. Finally, $d(\tilde{\omega} \wedge d\tilde{\eta}) = d\tilde{\omega} \wedge d\tilde{\eta} \pm \tilde{\omega} \wedge d^2\tilde{\eta} = d\tilde{\omega} \wedge d\tilde{\eta}$. $\square$

Prop. $H^\bullet$ is a contravariant functor.

Pf/ HW. $\square$