Math 411

Last time $\Lambda V^* = \bigoplus_{l \geq 0} \Lambda^l V^*$ is a graded anti-commutative k-algebra under the product
$$\Lambda^r V^* \times \Lambda^s V^* \longrightarrow \Lambda^{r+s} V^* .$$
$$(\omega, \eta) \longmapsto \omega \wedge \eta$$

The operation $\Lambda : V \to \Lambda V^*$ is a contravariant functor from the category of vector spaces$/k$ to graded anti-commutative k-algebras

Meaning:

$$\begin{tikzcd} U \arrow[r] & V \arrow[dl] \\ W \arrow[u] & \end{tikzcd} \xrightarrow{\ \Lambda\ } \begin{tikzcd} \Lambda U^* \arrow[dr] & \Lambda V^* \arrow[l] \\ & \Lambda W^* \arrow[u] \end{tikzcd}$$

functor: preserves commutative diagrams and takes id to id.

contravariant: reverses arrows.

Cartan derivative (exterior differentiation)

M manifold, $\Omega^k M$ = k-forms on M

Theorem. $\exists!$ sequence of linear maps

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \Omega^3 M \to \cdots \quad \Big\} \text{ de Rham complex}^{\star}$$

such that

1) If $f \in \Omega^0 M$, i.e. $f: M \to \mathbb{R}$, then df is the normal differential ($df_p : T_pM \to T_p\mathbb{R} \cong \mathbb{R} \Rightarrow df_p \in T_p^* M \Rightarrow df : M \to T^*M$, $p \mapsto df_p$, so $df \in \Omega^1 M$).

2) $d^2 := d \circ d = 0$ ($\star$ is a complex).

3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for $\omega \in \Omega^r M$ (product rule).

Main idea behind the proof of the theorem:

Locally, (1) – (3) force d to take the form

$$d\Big(\sum_\mu a_\mu \underbrace{dx_\mu}_{dx_{\mu_1}\wedge\cdots\wedge dx_{\mu_k}}\Big) = \sum_\mu \sum_{i=1}^{n} \frac{\partial a_\mu}{\partial x_i}\, dx_i \wedge dx_\mu .$$

This local expression behaves nicely under changes of coordinates, hence glues together to give a global map $d : \Omega^k M \to \Omega^{k+1} M$.

Example $M = \mathbb{R}^3$

$$\begin{aligned} d(x^2 y\, dx \wedge dz + x^2 z\, dy \wedge dz) &= (2xy\,dx + x^2\,dy) \wedge dx \wedge dz + (2xz\,dx + x^2\,dz) \wedge dy \wedge dz \\ &= (-x^2 + 2xz)\, dx \wedge dy \wedge dz . \end{aligned}$$

Prop. Defining d as above, $d^2 = 0$.

Pf/ This is a local question. Taking coordinates,

$$d^2(a\, dx_\mu) = d\left(\sum_j \frac{\partial a}{\partial x_j} dx_j \wedge dx_\mu\right)$$

$$= \sum_{i,j} \frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_\mu .$$

Then, if $i=j$, we have $dx_i \wedge dx_j = 0$ and if $i \neq j$ then

$$\frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j = -\frac{\partial^2 a}{\partial x_j \partial x_i} dx_j \wedge dx_i . \qquad \square$$

Lemma. $d(dw_1 \wedge \cdots \wedge dw_k) = 0$ for any forms $w_1, \dots, w_k$.

Pf/ We prove this by induction on k. For $k=1$, we have $d^2 w_1 = 0$. For $k > 1$, we get

$$d(dw_1 \wedge \cdots \wedge dw_k) = d^2 w_1 \wedge (dw_2 \wedge \cdots \wedge dw_k) \pm dw_1 \wedge d(dw_2 \wedge \cdots \wedge dw_k) = 0$$

by the product rule (which is easy to establish from the local expression for d) and induction. $\square$

Prop. Exterior differentiation commutes with pullbacks: if $f : M \to N$ is a mapping of manifolds, and $\omega \in \Omega^k N$, then

$$f^*(d\omega) = d(f^*\omega). \qquad \left(\begin{tikzcd} \Omega^k M \arrow[d, "d"'] & \Omega^k N \arrow[l, "f^*"'] \arrow[d, "d"] \arrow[dl, phantom, "///"] \\ \Omega^{k+1} M & \Omega^{k+1} N \arrow[l, "f^*"'] \end{tikzcd}\right)$$

Pf/ This is a local question. Let (U,h) be a chart at $p \in M$ and (V,k) a chart at $f(p) \in N$ such that $f(U) \subseteq V$. We may assume ω has the local expression $a(y)\, dy_{\mu_1} \wedge \cdots \wedge dy_{\mu_k}$. Then

$$f^*(d(a\, dy_\mu)) = f^*\left(\sum_{i=1}^n \frac{\partial a}{\partial y_i}\, dy_i \wedge dy_\mu \right) = \sum_{i=1}^n \frac{\partial a}{\partial y_i} \circ f \; df_i \wedge \underbrace{df_\mu}_{df_{\mu_1} \wedge \cdots \wedge df_{\mu_k}}$$

$$= \sum_{i=1}^n \sum_{j=1}^m \frac{\partial a}{\partial y_i} \circ f \; \frac{\partial f_i}{\partial x_j}\, dx_j \wedge df_\mu$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial a}{\partial y_i} \circ f \, \frac{\partial f_i}{\partial x_j} \, dx_j \wedge df_\mu$$

$$= \sum_{j=1}^{m} \frac{\partial (a \circ f)}{\partial x_j} \, dx_j \wedge df_\mu$$

$$= d\left(a \circ f \, df_\mu \right)$$

$$= d\left(f^*(a \, dy_\mu) \right). \quad \square$$

Consequence The proposition says we have a contravariant functors

1 Manifolds $\longrightarrow$ chain complexes

$$M \longmapsto \left(0 \to \Omega^0 M \to \Omega^1 M \to \Omega^2 M \to \cdots \right)$$

If $M \to N$, then we get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N & \to & \cdots \\
 & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\
0 & \to & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M & \to & \cdots
\end{array}$$

2 Manifolds $\longrightarrow$ Graded, anti-commutative, differential, algebras

$$M \longmapsto \Omega^{\bullet} M := \bigoplus_{l \geq 0} \Omega^{l} M$$