

Def. Let $\pi : T \rightarrow B$ be a surjection. A **section** of π (or of T) is a mapping $s : B \rightarrow T$ such that $\pi \circ s = \text{id}_B$.

VECTOR FIELDS

Def. A section of the tangent bundle $TM \xrightarrow{\pi} M$ (differentiable, of course) is called a **vector field** on M .

Remarks

1. Let $s : M \rightarrow TM$ be a vector field on M . The **zero locus** of s is $\{p \in M : s(p) = 0\}$. Note that this is a well-defined concept, i.e. $s(p)$ being zero does not depend on the chart.

(Recall if (U, h) is a chart at p on M , we get a standard chart on TM by

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\hat{h}} & h(U) \times \mathbb{R}^n \\ v \in T_q M & \longmapsto & (h(q), v(u, h)) \end{array}$$

If $v(u, h) = 0$, then $v(v, k) = 0 \forall$ charts (V, k) at q .

2. Let $S^2 = 2$ -sphere. If $TS^2 \cong S^2 \times \mathbb{R}^2$ as manifolds, ②
 then TS^2 would have a non-vanishing vector field. For instance,
 define $s: S^2 \rightarrow S^2 \times \mathbb{R}^2$. However, a famous theorem

$$p \mapsto (p, (1, 0))$$

 from topology (the hairy ball theorem) says there is no non-vanishing
 vector field on S^2 .

k-forms

Def. A k-form on a manifold is a section of $\Lambda^k T^*M$

The vector space of k-forms is denoted $\Omega^k M$.

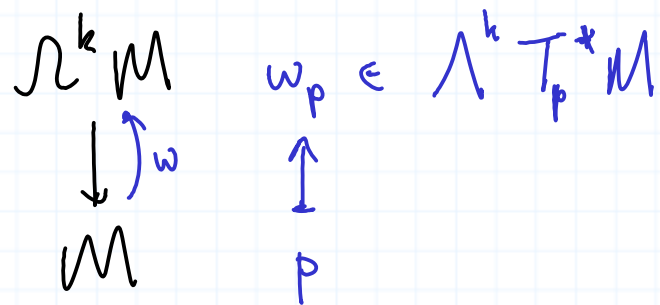
In coordinates: Let (U, h) be a chart on M .

Then $T_p M \xrightarrow{\cong} \mathbb{R}^n$ for each $p \in U$, gives a basis
 $v \in T_p M \mapsto v(U, h)$

$\left(\frac{\partial}{\partial x_i}\right)_p$ for $T_p M$, a dual basis $dx_{i,p} := \left(\frac{\partial}{\partial x_i}\right)_p^*$ for $T_p^* M$

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and a basis $\{ dx_{\mu_1, p} := dx_{\mu_1} \wedge \dots \wedge dx_{\mu_k} : 1 \leq \mu_1 < \dots < \mu_k \leq n \}$
 for $\Lambda^k T_p^* M$. Consider a k -form ω :



In local coordinates we get $\omega(p) = \sum_{\mu} w_{\mu}(p) dx_{\mu} = \sum_{\mu} w_{\mu}(p_1, \dots, p_n) dx_{\mu_1} \wedge \dots \wedge dx_{\mu_k}$

where each function $w_{\mu}: h(U) \rightarrow \mathbb{R}$ is differentiable.

Mappings / Pullbacks

Let $f: M \rightarrow \mathbb{R}$ on $p \in M$. Take coords. (U, h) at p on M and (\mathbb{R}, id) on \mathbb{R} .

We have a push forward mapping:

(★)

$$\begin{array}{ccc}
 \left(\frac{\partial}{\partial x_i}\right)_p \in T_p M & \xrightarrow{f_{*,p} = df_p} & T_{f(p)} \mathbb{R} \\
 \downarrow & & \downarrow \cong \\
 \mathbb{R}^n & \xrightarrow{\nabla f(p)} & \mathbb{R} \\
 e_i & \xrightarrow{\hspace{10em}} & \frac{\partial f}{\partial x_i}(p) := \frac{\partial (k \circ f \circ h^{-1})}{\partial x_i}(h(p)) \quad (k = \text{id, here})
 \end{array}$$

Thus, $df_p : T_p M \rightarrow \mathbb{R}$, i.e. $df_p \in T_p^* M$. In coords,

$$df_p = \sum_{j=1}^n \alpha_j(p) dx_{j,p} \quad \text{where } \alpha_i(p) \text{ is determined by}$$

$$df_p \left(\left(\frac{\partial}{\partial x_i}\right)_p \right) = \sum_j \alpha_j(p) dx_{j,p} \left(\left(\frac{\partial}{\partial x_i}\right)_p \right) = \alpha_i(p).$$

$\frac{\partial f}{\partial x_i}(p)$ (From ★, above) Thus, ★ $df_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) dx_{i,p}$.

Example on \mathbb{R}^2 :
 $d(xy^2 + y) = y^2 dx + (2xy + 1) dy$