 (Start with a review. See p.7 of lecture 10.)

Def. Let $\pi : T \to B$ be a surjection. A section of π (or of T) is a mapping $s : B \to T$ such that $\pi \circ s = id_B$.

VECTOR FIELDS

Def. A section of the tangent bundle $TM \underset{s}{\overset{\pi}{\rightleftarrows}} M$ (differentiable, of course) is called a vector field on M.

Remarks

1. Let $s : M \to TM$ be a vector field on M. The zero locus of s is $\{p \in M : s(p) = 0\}$. Note that this is a well-defined concept, i.e. $s(p)$ being zero does not depend on the chart. (Recall if (U,h) is a chart at p on M, we get a standard chart on TM by

$$\pi^{-1}(U) \xrightarrow{\hat{h}} h(U) \times \mathbb{R}^n$$
$$v \in T_qM \longmapsto (h(q), v(U,h))$$

If $v(U,h) = 0$, then $v(V,k) = 0$ $\forall$ charts (V,k) at q.

2. Let S^2 = 2-sphere. If $TS^2 \cong S^2 \times \mathbb{R}^2$ as manifolds, then TS^2 would have a non-vanishing vector field. For instance, define

$$s: S^2 \to S^2 \times \mathbb{R}^2$$
$$p \mapsto (p, (1,0))$$

However, a famous theorem from topology (the hairy ball theorem) says there is no non-vanishing vector field on S^2.

k-forms

Def. A k-form on a manifold is a section of $\Lambda^k T^* M$

The vector space of k-forms is denoted $\Omega^k M$.

In coordinates: Let (U,h) be a chart on M.

Then $T_pM \xrightarrow{\cong} \mathbb{R}^n$ for each $p \in U$, gives a basis
$v \in T_pM \mapsto v(U,h)$

$\left(\frac{\partial}{\partial x_i}\right)_p$ for T_pM, a dual basis $dx_{i,p} := \left(\frac{\partial}{\partial x_i}\right)_p^*$ for T_p^*M

and a basis $\{ dx_{\mu,p} := dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_k} : 1 \leq \mu_1 < \cdots < \mu_k \leq n \}$ for $\Lambda^k T_p^* M$. Consider a k-form ω:

$$\begin{array}{ccc} \Omega^k M & & \omega_p \in \Lambda^k T_p^* M \\ \downarrow \Big) \omega & & \uparrow \\ M & & p \end{array}$$

In local coordinates we get $\omega(p) = \sum\limits_{\mu} \omega_\mu(p)\, dx_\mu = \sum\limits_{\mu} \omega_\mu(p_1, \dots, p_n)\, dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_k}$

where each function $\omega_\mu : h(U) \to \mathbb{R}$ is differentiable.

Mappings / Pullbacks

Let $f : M \to \mathbb{R}$ on $p \in M$. Take coords. (U, h) at p on M and $(\mathbb{R}, id)$ on $\mathbb{R}$.

We have a push forward mapping:

$$(\star)\qquad \begin{array}{ccccc} \left(\frac{\partial}{\partial x_i}\right)_p & T_pM & \xrightarrow{f_{*,p} = df_p} & T_{f(p)}\mathbb{R} & \\ \downarrow & \downarrow \wr\wr & & \wr\wr \downarrow & \\ & \mathbb{R}^n & \xrightarrow{\nabla f(p)} & \mathbb{R} & \\ e_i & \longmapsto & & \frac{\partial f}{\partial x_i}(p) := \frac{\partial k\circ f h^{-1}}{\partial x_i}(h(p)) & (k = id, \text{ here}) \end{array}$$

Thus, $df_p : T_pM \to \mathbb{R}$, i.e. $df_p \in T_p^*M$. In coords.,

$$df_p = \sum_{j=1}^n \alpha_j(p)\, dx_{j,p} \quad \text{where} \quad \alpha_i(p) \text{ is determined by}$$

$$df_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_j \alpha_j(p)\, dx_{j,p}\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \alpha_i(p).$$

$$\overset{\text{\rotatebox{90}{=}}}{\frac{\partial f}{\partial x_i}(p)} \quad \text{(From } (\star)\text{, above)}$$

Thus, $\star$ $df_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, dx_{i,p}$.

Example on $\mathbb{R}^n$:

$d(xy^2+y)$

$= y^2dx + (2xy+1)\,dy$