Math 411

* Go over the HW.

Multilinear Algebra

Let V, W be vector spaces over a field k.

Products $\quad V \times W = \{ (v,w) : v \in V, w \in W \}$ is a vector space via

$$\lambda (v,w) + (v',w') = (\lambda v + v', \lambda w + w')$$

The product satisfies the following universal property:

Given a v.s. X/k and mappings $f: X \to V$, $g: X \to W$, $\exists!\, h: X \to V \times W$ commuting with the corresponding projection mappings:

$$\begin{tikzcd} & X \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & V \times W \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V & & W \end{tikzcd}$$

$h(x) := (f(x), g(x))$

Coproducts

$V \oplus W := V \times W$ with the same linear structure

$V \oplus W$ satisfies the universal property:

$$\begin{tikzcd}
& X & \\
& V \oplus W \arrow[u, dashed, "\exists! h"'] & \\
V \arrow[ur] \arrow[uur, bend left, "f"] & & W \arrow[ul] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

Here, $V \to V \oplus W$, $W \to V \oplus W$

$v \mapsto (v,0)$, $w \mapsto (0,w)$

$$h(v,w) = f(v) + g(w).$$

Note: There is a difference between $\times$ and $\oplus$ when considering products of infinitely many vector spaces.

Tensors

$$V \otimes W = F(V,W) / T$$

where $F(V,W)$ is the free vector space$/k$ on the symbols $[v,w]$

and T is the subspace of $F(V,W)$ generated by:

for all $\alpha \in k$, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$,

$$T: \quad \begin{array}{l} [v_1+v_2, w] - [v_1, w] - [v_2, w] \\ [v, w_1+w_2] - [v, w_1] - [v, w_2] \\ [\alpha v, w] - \alpha[v, w] \\ [v, \alpha w] - \alpha [v, w] \end{array}$$

Denote the equivalence class of $[v, w]$ in $F[V,W]/T$ by $v \otimes w$.

Thus, $V \otimes W$ is the vector space generated by the symbols $v \otimes w$ with relations:

$$(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \qquad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$(\alpha v) \otimes w = v \otimes (\alpha w) = \alpha(v \otimes w).$$

An arbitrary element of $V \otimes W$ has the form

$$\sum_{i=1}^{\ell} \alpha_i \, v_i \otimes w_i$$

WARNING Say $v \in V$, $w \in W$ and $V \neq W$. Then $w \otimes v \notin V \otimes W$ so we can't have $v \otimes w = w \otimes v$.

Example $V = \mathbb{R}^2$ with std. basis e_1, e_2

$W = \mathbb{R}^3$ with std. basis f_1, f_2, f_3.

$$(1,2) \otimes (3,4,5) = (e_1 + 2e_2) \otimes (3f_1 + 4f_2 + 5f_3)$$

$$= 3\, e_1 \otimes f_1 + 4\, e_1 \otimes f_2 + 5\, e_1 \otimes f_3 + 6\, e_2 \otimes f_1 + 8\, e_2 \otimes f_2 + 10\, e_2 \otimes f_3 .$$

(In fact, a basis for $\mathbb{R}^2 \otimes \mathbb{R}^3$ is $\{e_i \otimes f_j\}_{\substack{i=1,2 \\ j=1,2,3}}$.)

Note: $e_1 \otimes f_1 + 2e_1 \otimes f_2 = e_1 \otimes (f_1 + 2f_2)$ but there are no $v \in \mathbb{R}^2$, $w \in \mathbb{R}^3$ such that $v \otimes w = e_1 \otimes f_1 + e_2 \otimes f_2$. (Try letting $v = \alpha_1 e_1 + \alpha_2 e_2$ and $w = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3$ and expanding $v \otimes w$.)