

# Math 411

\* Go over the HW.

## Multilinear Algebra

Let  $V, W$  be vector spaces over a field  $k$ .

### Products

$V \times W = \{ (v, w) : v \in V, w \in W \}$  is a vector space via

$$\lambda (v, w) + (v', w') = (\lambda v + v', \lambda w + w')$$

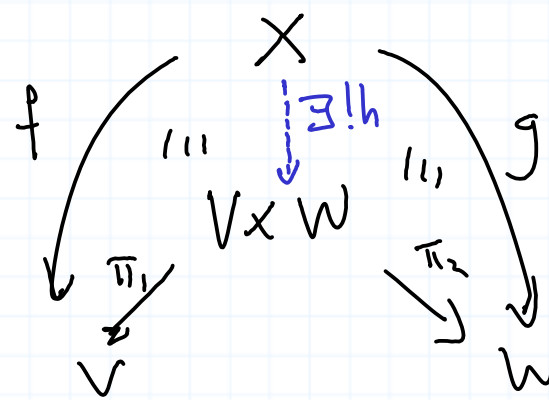
The product satisfies the following universal property:

Given a v.s.  $X/k$

and mappings  $f: X \rightarrow V, g: X \rightarrow W,$

$\exists! h: X \rightarrow V \times W$  commuting with

the corresponding projection mappings:

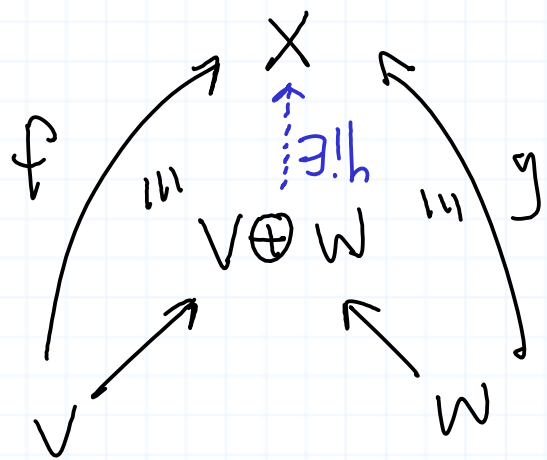


$$h(x) := (f(x), g(x))$$

## Coproducts

$V \oplus W := V \times W$  with the same linear structure

$V \oplus W$  satisfies the universal property:



Here,  $V \rightarrow V \oplus W$ ,  $W \rightarrow V \oplus W$   
 $v \mapsto (v, 0)$   $w \mapsto (0, w)$

$$h(v, w) = f(v) + g(w).$$

Note: There is a difference between  $\times$  and  $\oplus$  when considering products of infinitely many vector spaces.

## Tensors

$$V \otimes W = \frac{F(V, W)}{T}$$

where  $F(V, W)$  is the free vector space/ $k$  on the symbols  $[v, w]$

and  $T$  is the subspace of  $F(V, W)$  generated by:

for all  $\alpha \in k$ ,  $v_1, v_2 \in V$ , and  $w_1, w_2 \in W$ ,

$$\begin{aligned} T: \quad & [v_1 + v_2, w] = [v_1, w] + [v_2, w] \\ & [v, w_1 + w_2] = [v, w_1] + [v, w_2] \\ & [\alpha v, w] = \alpha [v, w] \\ & [v, \alpha w] = \alpha [v, w] \end{aligned}$$

Denote the equivalence class of  $[v, w]$  in  $F[V, W]/T$  by  $v \otimes w$ .

Thus,  $V \otimes W$  is the vector space generated by the symbols  $v \otimes w$  with relations:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$(\alpha v) \otimes w = v \otimes (\alpha w) = \alpha (v \otimes w).$$

An arbitrary element of  $V \otimes W$  has the form

$$\sum_{i=1}^l \alpha_i v_i \otimes w_i$$

**WARNING**  
Say  $v \in V, w \in W$   
and  $V \neq W$ . Then  
 $w \otimes v \notin V \otimes W$  so we  
can't have  $v \otimes w = w \otimes v$ .

**Example**  $V = \mathbb{R}^2$  with std. basis  $e_1, e_2$   
 $W = \mathbb{R}^3$  with std. basis  $f_1, f_2, f_3$ .

$$\begin{aligned} (1, 2) \otimes (3, 4, 5) &= (e_1 + 2e_2) \otimes (3f_1 + 4f_2 + 5f_3) \\ &= 3e_1 \otimes f_1 + 4e_1 \otimes f_2 + 5e_1 \otimes f_3 + 6e_2 \otimes f_1 + 8e_2 \otimes f_2 + 10e_2 \otimes f_3. \end{aligned}$$

(In fact, a basis for  $\mathbb{R}^2 \otimes \mathbb{R}^3$  is  $\{e_i \otimes f_j\}_{\substack{i=1,2 \\ j=1,2,3}}$ .)

Note:  $e_1 \otimes f_1 + 2e_1 \otimes f_2 = e_1 \otimes (f_1 + 2f_2)$  but there are no  $v \in \mathbb{R}^2, w \in \mathbb{R}^3$   
such that  $v \otimes w = e_1 \otimes f_1 + e_2 \otimes f_2$ . (Try letting  $v = \alpha_1 e_1 + \alpha_2 e_2$   
and  $w = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3$  and expanding  $v \otimes w$ .)