Reality check. Math 411

A. Compute the derivatives of the following functions at the given points as linear mappings.

1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

   $(x,y) \longmapsto (x^2-y,\ xy,\ x^3+2xy^2)$ at $p=(1,1)$

2. $\alpha : (-1,1) \longrightarrow \mathbb{R}^3$

   $t \longmapsto (t,\ \cos(t),\ \sin(t))$ at $t=0$

3. $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$

   $(x,y,z) \longmapsto x+y^2+z^3$ at $p=(1,1,1)$.

Solutions:

1. $f'(1,1)(x,y) = (2x-y,\ x+y,\ 5x+4y)$.
2. $\alpha'(0)(t) = (t,\ 0,\ t)$.
3. $g'(1,1,1)(x,y,z) = x+2y+3z^2$.

B.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^k \to \mathbb{R}^m$. State the chain rule for f and g at the point $p \in \mathbb{R}^n$ in terms of Jacobian matrices and in terms of linear functions.

Solution:

$$J(g \circ f)(p) = J_g(f(p))\, Jf(p)$$

$$(g \circ f)'(p) = g'(f(p)) \circ f'(p).$$

2. Suppose $w : \mathbb{R}^n \to \mathbb{R}$. Show that for all $x \in \mathbb{R}^n$,

$$\frac{d}{dt} w(tx) = \sum_{j=1}^{n} x_j \frac{\partial w}{\partial x_j}(tx).$$

Pf/ Define $\alpha : \mathbb{R} \to \mathbb{R}^n$ by $\alpha(t) = tx$. $= (tx_1, \dots, tx_n)$ $\Rightarrow J\alpha(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. By the chain rule,

$$\left[\frac{d}{dt} w(tx)\right] = J(w \circ \alpha)(t) = Jw(\alpha(t))\, J\alpha'(t) = Jw(tx)\, J\alpha'(t)$$

$$= \begin{bmatrix} \frac{\partial w}{\partial x_1}(tx) & \cdots & \frac{\partial w}{\partial x_n}(tx) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} x_j \frac{\partial w}{\partial x_j}(tx). \quad \square$$

Tangent space (continued from last time)

T_pM is an n-dimensional vector space $/\mathbb{R}$. This might be most easily see via $T_p^{phys}(M)$: Choose a chart (U,h) at p.

Thm

$$\begin{aligned} T_p^{phys}(M) &\longrightarrow \mathbb{R}^n \\ v &\longmapsto v(U,h) \end{aligned} \qquad (\star)$$

gives an isomorphism of vector spaces. Note: the linear structure on $T_p^{phys}(M)$ is given by $v,w \in T_p^{phys}(M) \Rightarrow \lambda v + w \in T_p^{phys}(M)$ where $\lambda v + w : D_pM \longrightarrow \mathbb{R}^n$ by $(\lambda v + w)(V,k) := \lambda v(V,k) + w(V,k)$.

The key to $(\star)$ is that knowing $v(U,h)$ for any particular chart determines v for all charts at p.

Having fixed a chart (U, h) at p, the standard basis for $T_p M$ denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ is defined as follows:

As elements of $T_p^{alg}(M)$, i.e., as derivations, let f be a germ at p. Then

$$\left(\frac{\partial}{\partial x_i}\right)_p f := \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p))$$

for any chart (U, h) at p. Hence, $\left(\sum a_i \left(\frac{\partial}{\partial x_i}\right)_p\right)(f) = \sum a_i \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p))$.

As elements of $T_p^{phys} M$, we have $\left(\frac{\partial}{\partial x_i}\right)_p (U, h) = e_i = i^{th}$ standard basis vector. Of course, the notions are compatible via our correspondence

$$T_p^{phys}(M) \longleftrightarrow T_p^{alg}(M).$$

Tangent Map / Differential

Let $f: M \to N$ be a mapping of manifolds, and let $p \in M$.
We get an induced linear mapping

$$df_p : T_p M \longrightarrow T_{f(p)} N$$

Three versions:

geometric – α a curve in M $\longmapsto$ $df_p(\alpha) := f \circ \alpha$
$\alpha(0) = p$ $\qquad (f \circ \alpha)(0) = f(p)$.

algebraic – $v : \mathcal{E}_p(M) \to \mathbb{R}$ $\longmapsto$ $df_p(v) : \mathcal{E}_{f(p)}(N) \to \mathbb{R}$
derivation of germs at p $\qquad g \mapsto v(g \circ f)$

physical – $v \in T_p^{phys}(M)$ $\longmapsto$ $df_p(v) : D_{f(p)} N \longrightarrow \mathbb{R}^n$
$v : D_p(M) \to \mathbb{R}^m$ $\qquad (V, k) \mapsto (k \circ f \circ h^{-1})'_{h(p)} \big(v(U, h)\big)$.

(see next page)

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ h \downarrow & & \downarrow k \\ \mathbb{R}^m \supseteq h(U) & \xrightarrow{kfh^{-1}} & k(V) \subseteq \mathbb{R}^n \end{array}$$

Given the chart (V, k), choose any chart (U, h) with $U \subseteq f^{-1}(V)$.

open since f is cts.

Exercise: $f : \mathbb{P}^2 \to \mathbb{P}^3$

$(x,y,z) \mapsto (x^3, y^3, z^3, xyz)$

Let $p = (1, s, t) \in U_x$. Since $f(1,s,t) = (1, s^3, t^3, st)$, consider the standard open set $V = \{(a,b,c,d) \in \mathbb{P}^3 : a \neq 0\}$ with $\varphi_V(a,b,c,d) = (\frac{b}{a}, \frac{c}{a}, \frac{d}{a})$.

Describe df_p as a mapping of physical tangent spaces in terms of these charts.

i.e. $\varphi_V \circ f \circ \varphi_x^{-1}(s,t) = (s^3, t^3, st)$

Solution: With respect to these charts, f becomes the mapping $\tilde{f}(u,v) = (s^3, t^3, st)$. Then $J\tilde{f}(s,t) = \begin{bmatrix} 3s^2 & 0 \\ 0 & 3t^2 \\ t & s \end{bmatrix}$.

So given an element of $v \in T_p^{phys} M$ that assigns the vector (v_1, v_2, v_3) to $(U_x, \mathfrak{d}_x)$, we have that $df_p(v)$ assigns

$$Jf(s,t) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{to} \quad (V, \mathfrak{d}_V).$$