HW 7, due Friday, March 29 *Math 411*

For this assignment, please refer to section 4 of our handout on topology, available from our website.

1. Explain why the components and the path components of a manifold are the same. (Quote the right result from the handout, and argue that it applies.)

2. Let X be a topological space.

   (a) Suppose X is not connected. Show that there exist nonempty, disjoint subsets A and B of X such that both A and B are open and $X = A \cup B$.

   (b) Let $Y \subset X$ with the subspace topology. Suppose X is not connected, and write $X = A \cup B$, as in the previous exercise. If Y is connected, show that Y is contained in either A or B.

3. Let X and Y be topological spaces, and let $f\colon X \to Y$ be any mapping (of sets). We say f is *locally constant* if for each $x \in X$ there exists an open neighborhood U of x such that f restricted to U is constant.

   (a) Prove that if f is locally constant, it is continuous.

   (b) Prove that if f is locally constant, then it is constant on each connected component of X.

   (c) Let M be a manifold. Prove that $H^0 M \approx \mathbb{R}^c$ where c is the number of connected components of M.

4. Let $f\colon M \to N$ be a mapping of manifolds.

   (a) Prove that for each $k \geq 0$, the pullback $f^*\colon \Omega^k N \to \Omega^k M$ induces a well-defined mapping $f^*\colon H^k N \to H^k M$.

   (b) Prove that if f is constant, then $f^*\colon H^k N \to H^k M$ is the zero map if $k > 0$.

   (c) Prove that if f is constant and both M and N are connected, then $f^*\colon H^0 N \to H^0 M$ is an isomorphism. (Note: $H^0 M \approx H^0 N \approx \mathbb{R}$ in this case.)

5. Suppose M is a contractible manifold and h is a homotopy between the identity mapping on M and a constant mapping. Let $\omega \in \Omega^k M$ be a cocycle, i.e., $d\omega = 0$. We saw in class that if $k \geq 1$, then $\omega = dP(h^*\omega)$ where P is the prism operator.

   Let F be vector field on $\mathbb{R}^3$ with curl $F = 0$. Calculate $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ such that grad $\phi = F$ by calculating $P(h^*\omega)$ where ω is a suitably defined 1-form on $\mathbb{R}^3$ and $h(x, y, z) =$

(tx, ty, tz) for $t \in [0, 1]$ is a homotopy of the zero-mapping and the identity on $\mathbb{R}^3$. Show that for each $p \in \mathbb{R}^3$,

$$\phi(p) = \int_\gamma F \cdot d\vec{t},$$

the flow of F along γ where $\gamma(t) := tp$ for $t \in [0, 1]$.

6. A *Lie group* is a manifold G with a group structure so that both the multiplication and inverse mappings:

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth. Let G be a connected Lie group, and let U be an open neighborhood of the identity. Show that U generates G. (Define $U^n = \{g_1 \cdots g_n : g_1, \ldots, g_n \in U\}$. You must show that $\cup_{n \geq 1} U^n = G$.)