HW 3, due Friday, February 22 *Math 411*

1. (a) Let X and Y be sets. What are $X \times Y$ and $X \oplus Y$? Show they satisfy the appropriate universal properties.

   (b) Let X and Y be objects in any category. If A and B are both products of X and Y, use the universal property of products to show that A and B are isomorphic.

2. Let V be a vector space over a field K, and let $v_1, \ldots, v_n$ be a basis for V. For $i = 1, \ldots, n$, define mappings $v_i^*\colon V \to K$ by requiring $v_i^*(v_j) = \delta(i,j)$ and extending linearly. Show that $v_1^*, \ldots, v_n^*$ is a basis for $V^* = \hom(V, K)$.

3. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. The function L is represented by the matrix A whose j-th column is $L(e_j)$, the image of the j-th standard basis vector. Choosing the basis dual to the standard basis, show that the matrix representing $L^*\colon (\mathbb{R}^m)^* \to (\mathbb{R}^n)^*$ is A^t, the transpose of A.

4. The n-sphere is the set $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. It has a topology induced by $\mathbb{R}^{n+1}$, i.e., a set is open in S^n iff it is the intersection of an open set of $\mathbb{R}^{n+1}$ with S^n. Each point in S^n has some non-zero coordinate. For $i = 1, \ldots, n+1$, define $U_i^+ = \{x \in \ S^n : x_i > 0\}$ and $U_i^- = \{x \in \ S^n : x_i < 0\}$. Define $\pi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$. Then the collection of charts (U_i^+, π_i) and (U_i^-, π_i) for $i = 1, \ldots, n+1$ serves as an atlas for S^n.

   (a) Show that the charts constructed in this way are differentially compatible.

   (b) Consider the function $f(x) = \sum_{i=1}^{n+1} x_i^4$ defined on S^n. Let $p \in S^n$ with $p_1 > 0$. With respect to the chart (U_1^+, π_1), take the local coordinates to be $x_2, \ldots, x_{n+1}$, as seems natural in this case. Let
$$v := a\left(\frac{\partial}{\partial x_2}\right)_p + b\left(\frac{\partial}{\partial x_3}\right)_p$$
   be a tangent vector at p. Calculate $v(f)$. In other words, think of v as a derivation and apply it to f.

5. Let $V = W = \mathbb{R}^3$ and let $L\colon V \to W$ be a linear mapping with matrix $A = (a_{ij})$ relative to the standard bases bases v_1, v_2, v_3 for V and w_1, w_2, w_3 for W. Of course, $v_i = w_i = e_i$ for all i, but it is useful to have separate notation. Let
$$\omega = w_1^* \wedge w_3^* \in \Lambda^2 W^*.$$

   (a) Corresponding to ω is a bilinear, alternating form $\tilde{\omega} \in \mathrm{Alt}^2 W$:
$$\tilde{\omega}\colon W \times W \to \mathbb{R}.$$
   Describe $\tilde{\omega}$ by calculating the images of (w_i, w_j) for $1 \le i < j \le 3$.

(b) Find α, β, γ as functions of the entries of A so that $L^*\omega = \alpha\, v_1^* \wedge v_2^* + \beta\, v_1^* \wedge v_3^* + \gamma\, v_2^* \wedge v_3^*$. Do this by computing $(L^*\omega)(v_i \wedge v_j)$ for each $i < j$.

(c) The mapping L is given by

$$L(x, y, z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z).$$

Calculate

$$(a_{11}v_1^* + a_{12}v_2^* + a_{13}v_3^*) \wedge (a_{31}v_1^* + a_{32}v_2^* + a_{33}v_3^*)$$

and compare with your answer to the previous problem.