HW 1, due Friday, February 8 *Math 411*

1. Prove the first clause of Prop. 1.7 in the topology handout.

2. Prove Prop. 1.9 in the topology handout.

3. Give the simplest example of a topological space that is not Hausdorff.

4. Let X and Y be disjoint copies of $\mathbb{R}$ with the usual topology. For $x \in X$ and $y \in Y$, say $x \sim y$ if $x = y \neq 0$ as real numbers. Define $L = (X \cup Y)/\sim$ as a topological space with the quotient topology. In other words, if

$$\begin{aligned} \pi\colon X \cup Y &\to L \\ a &\mapsto [a], \end{aligned}$$

where $[a] = \{b \in X \cup Y : a \sim b\}$, then a subset $U \subseteq L$ is open if and only if $\pi^{-1}(U)$ is open in $X \cup Y$ (which means $\pi^{-1}(U) \cap X$ and $\pi^{-1}(U) \cap Y$ are open a subsets of the real numbers). Thus, L is essentially the real number line with two origins, 0_X from X and 0_Y from Y. Show that L is locally Euclidean but not Hausdorff.

5. Recall that $\mathbb{P}^2$ is the set of one-dimensional vector subspaces of $\mathbb{R}^3$ and a *line* in $\mathbb{P}^2$ is a two-dimensional subspace. Prove that two points in $\mathbb{P}^2$ determine a unique line and conversely, two lines determine a unique point.

6. Consider $\mathbb{P}^n$ with its standard atlas. Calculate the transition function from U_0 to U_1, i.e., $\phi_1 \circ \phi_0^{-1}$.

7. Suppose M is a manifold, let $f\colon M \to \mathbb{R}$, and let (U, h) and (V, k) be two charts about $p \in M$. Show that if f is differentiable at p relative to a chart (U, h), then f is differentiable at p relative to (V, k).

8. Let V be an n-dimensional vector space over $\mathbb{R}$. A choice of ordered basis $\mathbb{B}$, for V gives a linear isomorphism

$$h_{\mathbb{B}}\colon V \to \mathbb{R}^n.$$

Give V a topology by declaring a set open in V iff its image under $h_{\mathbb{B}}$ is open in $\mathbb{R}^n$. Then $(V, h_{\mathbb{B}})$ is a chart at each point in V, hence, $\mathfrak{A} := \{(V, h_{\mathbb{B}})\}$ is an atlas. The maximal atlas containing $\mathfrak{A}$ determines a differentiable structure on V and makes V a manifold. Show that the choice of a different ordered basis determines the same differentiable structure. In this sense, V has a canonical manifold structure.

9. Is $(-1, 1) \subset \mathbb{R}$ diffeomorphic to $\mathbb{R}$? Is the open disc of radius 1 in $\mathbb{R}^2$ diffeomorphic to $\mathbb{R}^2$? You may use the internet to help answer this question if you cannot figure it out on your own.