Review of Measure Theory

outer measure If $I \subseteq \mathbb{R}^n$ is a rectangle, let $\mu(I)$ = product of lengths of sides.
If $X \subseteq \mathbb{R}^n$, define $\mu(X) = \inf\{\sum_k \mu(I_k)\}$, taking the inf over covers $\{I_k\}$ of X by rectangles.

$\bigcup_k I_k \supseteq X$.

σ-algebra Let X be any set. Then $\Sigma \subseteq 2^X$ is a σ-algebra if

1. $\emptyset \in \Sigma$, $X \in \Sigma$
2. $A \in \Sigma \Rightarrow A^c \in \Sigma$ (complements)
3. $\{A_k\}_{k=1}^{\infty} \subseteq \Sigma \Rightarrow \bigcup_k A_k \in \Sigma$ (countable unions)

measure Let X be a set with σ-algebra Σ. A measure is a function $\mu : \Sigma \to \mathbb{R} \cup \{\infty\}$ s.t.

(1) $\mu(\emptyset) = 0$

(2) $0 \leq \mu(A) \leq \infty \quad \forall A \in \Sigma$

(3) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ if $\{A_k\} \subseteq \Sigma$ with the A_k pairwise disjoint.

Lebesgue measurable sets

$X \subseteq \mathbb{R}^n$ is Lebesgue measurable if $\forall A \subseteq \mathbb{R}^n$

$$\mu(A) = \mu(A \cap X) + \mu(A \cap X^c),$$

i.e. X "splits every set additively in measure".

Thm. The collection of Lebesgue measurable sets $\mathcal{L}$ forms a σ-algebra on $\mathbb{R}^n$ with measure μ (restricted to $\mathcal{L}$). $\mathcal{L}$ contains all open sets.

Measurable functions Let (X, Σ, μ) be a measure space. A function $f : X \to \mathbb{R}$ is measurable if $f^{-1}(a, \infty) \in \Sigma \quad \forall a \in \mathbb{R}$.

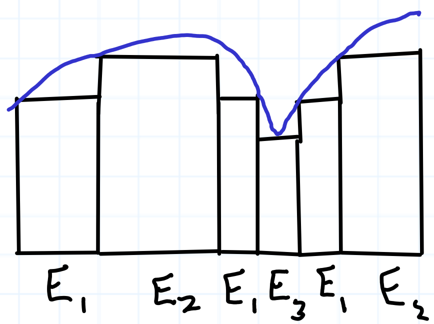
Integration (X, Σ, μ) a measure space. The **characteristic function** for $A \subseteq X$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For A measurable and $c \in \mathbb{R}$, define

$$\int c\chi_A = c\mu(A).$$

The integral. If $f(x) \geq 0$ $\forall x$, approximate f with "simple" functions $\varphi = \sum a_i \chi_{E_i}$ s.t. $0 \leq \varphi(x) \leq f(x)$ $\forall x$:



Approximating a function with a simple function

Define $\int_X f = \sup\left\{ \sum a_i \int_X \chi_{E_i} \right\}$, the sup over all $0 \le \overset{\sum a_i \chi_{E_i}}{\phi} \le f$,

If f is not necessarily nonnegative, write $f = f^+ - f^-$ where

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then f is integrable if $\int f^+$ and $\int f^-$ are finite, in which case

$$\int f := \int f^+ - \int f^- .$$