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ANALYTICAL DYNAMICS OF FIELDS

Introduction. Somewhat idiosyncratically, I like to distinguish the "classical mechanics" of particles from what (until about the time of the appearance—in 1904—of E. T. Whittaker's TREATISE ON THE ANALYTICAL DYNAMICS OF PARTICLES AND RIGID BODIES) used to be called "analytical mechanics." I take the former to embrace *all that can be said, by whatever formal means*, about the dynamics of particulate systems of all descriptions, but understand the latter terminology to refer specifically to the resources latent in the Lagrangian formalism, in the Hamiltonian formalism, in the Hamilton-Jacobi formalism, in their less-well-known companion formalisms (such, for example, as the Appell formalism) and their associated variational principles. Since it is easy to think of systems—particularly, but not exclusively, *dissipative* systems—to which none of the formalisms just ennumerated usefully pertain, it is clear that "analytical mechanics" is by nature a sub-division of a broader discipline. Remarkably, it is (or appears to be) within the confines of that sub-discipline that God prefers to frame His most fundamental utterances.

When one turns from the mechanics of spatially localized systems to the mechanics of *distributed* systems—i.e., from the dynamics of particles to the dynamics of fields—one encounters a similar situation. To think generally of "field theory" is to think of a subject so broad as to embrace all aspects of the motion of fluids, of elastic solids, of electromagnetic and gravitational fields, all—surprisingly—of the quantum mechanics of particles (at least formally), and of much else besides. Within that broad field lives the subject to which my chapter title refers.

Our strategy will be to look to a graded sequence of particulate systems, the limiting member of which will, by design, have in fact the character of a

field. By tracking the analytical mechanics of the individual members of the sequence we shall obtain the analytical mechanics of its limiting member—the "analytical mechanics of a field." The results thus achieved will be so strikingly simple as to admit readily and unproblematically of generalization. This mode of proceeding will, by the way, yield a general-purpose field-theoretic language which is automatically consonant with the analytical mechanics of particles, and is therefore preadapted to the discussion of particles and fields *in interaction*.

Dynamics of one-dimensional crystals. Take N identical particles of mass m, and $N+1$ identical springs of strength k, and form the "crystal" illustrated in the first of the following figures. The terminal springs are attached to "walls" which stand a distance ℓ from one another, so when the system is at rest each

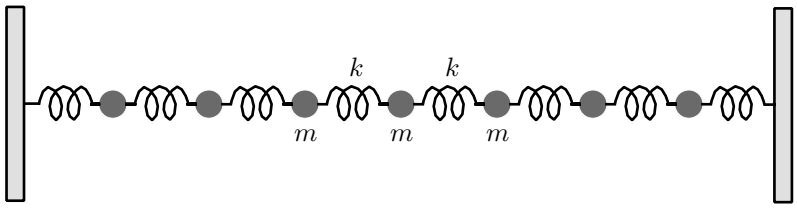


FIGURE 1: *Equilibrium configuration of an N-particle crystal with clamped boundaries.*

spring has length $a = \ell/(N+1)$. The system derives its "one-dimensionality" not so much from the linearity of its design as from the explicit stipulation that *transverse motion will be disallowed*. The allowed motion is longitudinal, which by natural orientation of a Cartesian coordinate system means "along the x-axis." If we associate the origin of the x-axis with the anchor point on the left, then we can write $x_n = na$ to describe the location of the n^{th} "atom" in a crystal at rest. Our dynamical assignment is to develop the functions $x_n(t)$ $(n = 1, 2, \ldots, N)$ which describe the successive locations of the constituent atoms in a crystal *not* at rest.

To that end, let us (see the following figure) introduce "excursion variables" φ_n by means of the equations $x_n = na + \varphi_n$. Evidently φ_n serves to describe

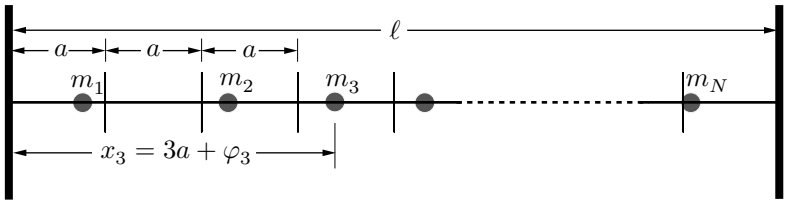


FIGURE 2: *Use of excursion variables to describe the instantaneous configuration of a crystal not at rest.*

the location of the n^{th} atom *relative to its equilibrium position*. It is in precisely the spirit of the familiar "theory of small oscillations" that, in our effort to

comprehend the motion of the crystal, we agree to look to the time-dependence of the variables φ_n.

We invested energy in the formation of our crystal; the interconnecting springs are, after all, stretched (else compressed) in the general case. Energy over and above that "ground state energy" must be invested if we wish to set the crystal in motion. That energy can evidently be described $E = T + U$ with

$$T = \tfrac{1}{2} m \left\{ \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \cdots + \dot{\varphi}_{N-1}^2 + \dot{\varphi}_N^2 \right\} \tag{1}$$

$$U = \tfrac{1}{2}\, k \, \left\{ \varphi_1^2 + (\varphi_2 - \varphi_1)^2 + \cdots + (\varphi_N - \varphi_{N-1})^2 + \varphi_N^2 \right\} \tag{2}$$

Forming the Lagrangian $L = T - U$ and working from

$$\left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{\varphi}_n} - \frac{\partial}{\partial \varphi_n} \right\} L = 0 \qquad\qquad n = 1, 2, \ldots, N \tag{3}$$

we readily obtain the following explicit equations of motion:

$$\left.\begin{aligned}
m\ddot{\varphi}_1 &= -k(\qquad\quad + 2\varphi_1 - \varphi_2 \quad) \\
m\ddot{\varphi}_2 &= -k(-\varphi_1 \quad + 2\varphi_2 - \varphi_3 \quad) \\
&\vdots \\
m\ddot{\varphi}_n &= -k(-\varphi_{n-1} + 2\varphi_n - \varphi_{n+1}) \qquad n = 2, 3, \ldots, N-1 \\
&\vdots \\
m\ddot{\varphi}_N &= -k(-\varphi_{N-1} + 2\varphi_N \qquad)
\end{aligned}\right\} \tag{4}$$

These comprise a coupled system of N 2^{nd}-order ordinary differential equations in N variables. Equations (4) are notable in particular for their *linearity*, which we might emphasize by writing

$$\ddot{\boldsymbol{\varphi}} + \mathbb{S}\boldsymbol{\varphi} = \mathbf{0} \tag{5}$$

where

$$\boldsymbol{\varphi} \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_N \end{pmatrix} \text{ and } \mathbb{S} \text{ has the structure } \mathbb{S} \equiv \Omega^2 \begin{pmatrix} a & b & & & & & \\ b & a & b & & & & \\ & b & a & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & b & a & b & \\ & & & & b & a & \end{pmatrix}$$

with $\Omega^2 = k/m$, $a = 2$ and $b = -1$. The design of the $\mathbb{S}$ matrix (in which all unreported matrix elements are zero) pretty clearly and directly mimics the physical design of the crystal itself.

We have now before us a physical system with a long and important history, an interesting brief account of which can be found in the opening sections of

L. Brillouin's classic *Wave Propagation in Periodic Structures* (1946). The first chapter in that history was written by Newton himself, who used a one-dimensional crystal to model an air column in his pioneering attempt to compute the velocity of sound. The system continues to serve as a point of entry into the study of real crystals (solid state physics), of the vibration of molecules, of transmission lines and of much else. And the system gives rise to an analytical problem of rich methodological interest. It is to aspects of the latter that I am motivated now to give brief attention.

Suppose we return to (5) with the *assumption* that the atoms oscillate with possibly distinct amplitudes A_n but in perfect synchrony:

$$\boldsymbol{\varphi}(t) = \boldsymbol{A}\, e^{i\omega t}$$

Immediately

$$(\mathbb{S} - \omega^2 \mathbb{I}\,)\boldsymbol{A} = \boldsymbol{0}$$

Evidently ω^2 must be set equal to one or another of the eigenvalues of $\mathbb{S}$, i.e. to one or another of the zeroes of the characteristic polynomial

$$\det(\mathbb{S} - \lambda\mathbb{I}) = s_0 + s_1\lambda + s_2\lambda^2 + \cdots + s_N\lambda^N$$

and $\boldsymbol{A}$ must be proportional to the corresponding eigenvector. Familiarly, the real symmetry of $\mathbb{S}$ is by itself sufficient to insure (i) the reality of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ and (ii) the orthogonality of the eigenvectors: $\boldsymbol{A}_m \cdot \boldsymbol{A}_n = 0$ if $m \neq n$ which (after normalization) can be notated $\boldsymbol{A}_m \cdot \boldsymbol{A}_n = \delta_{mn}$. This is valuable information, but not in itself sufficient to provide *explicit descriptions* of the eigenvalues/vectors. It is, in particular, not in itself sufficient to establish that all eigenvalues are necessarily *non-negative*. Proceeding therefore (for the moment) formally, we note that the linearity of the equations of motion (5) carries with it a "principle of superposition," and are led to write

$$\begin{aligned}\boldsymbol{\varphi}(t) &= \sum_{n=1}^{N} \alpha_n \boldsymbol{A}_n e^{+i\omega_n t} + \sum_{n=1}^{N} \beta_n \boldsymbol{A}_n e^{-i\omega_n t} \\ &= \text{superposition of "normal modes"}\end{aligned}$$

with $\omega_n = \sqrt{\lambda_n}$. The complex numbers α_n and β_n are fixed by imposition of the requirements that $\boldsymbol{\varphi}(t)$ be real and that it conform to the prescribed initial data $\boldsymbol{\varphi}(0)$ and $\dot{\boldsymbol{\varphi}}(0)$; the orthonormality of the eigenvectors greatly simplifies the computational labor at this point.

The program sketched above is in fact not at all specific to crystals, but pertains generally to the "theory of small oscillations," i.e., to the classical motion of *all* particulate systems as they jiggle harmonically about points of stable equilibrium. And it admits of a great variety of alternative formulations, the relative utility of which depends upon particular features of the system in hand, and the nature of the questions uppermost in one's mind. Of these, I must be content here to sketch only one:[1]

[1] For a fairly elaborate review of the formal possiblities, see Chapters I and II of my Classical Theory of Fields (1965).

Proceeding very much in the spirit of Hamilton, let us agree to promote $\dot{\boldsymbol{\varphi}}$ to the status of an independent variable, writing $\dot{\boldsymbol{\varphi}} \equiv \boldsymbol{\chi}$. In place of (5) we can then write

$$\dot{\boldsymbol{\Phi}} = \mathbb{W}\boldsymbol{\Phi} \tag{6}$$

with

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\chi} \end{pmatrix} \qquad \text{and} \qquad \mathbb{W} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{S} & \mathbb{O} \end{pmatrix}$$

Whereas (5) is a coupled system of N differential equations of 2^{nd} order, (6) is a $2n$-fold system of 1^{st} order, as admits therefore immediately of formal solution:

$$\boldsymbol{\Phi}(t) = e^{\mathbb{W}t}\boldsymbol{\Phi}(0)$$

The problem at this point is to assign explicit meaning to the matrix $e^{\mathbb{W}t}$. It is sometimes possible to gain useful information directly from the expansion

$$e^{\mathbb{W}t} = \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathbb{W}t)^n$$

but I propose to sketch an alternative mode of approach. By the Cauchy integral theorem

$$\tfrac{1}{2\pi i}\oint_C \tfrac{1}{z-W}e^{zt}dz = e^{Wt}$$

where C is any closed contour which envelops the singularity (simple pole) at $z = W$. We therefore expect to have

$$e^{\mathbb{W}t} = -\tfrac{1}{2\pi i}\oint_C \mathbb{R}(z)e^{zt}dz \quad \text{with} \quad \mathbb{R}(z) \equiv (\,\mathbb{W} - z\mathbb{I}\,)^{-1}$$

where C is any contour which envelops the *spectrum* of $\mathbb{W}$—the set of z-values at which $\mathbb{W} - z\mathbb{I}$ fails to be invertible. Such z-values are, of course, precisely the eigenvalues of $\mathbb{W}$. This mode of proceeding becomes useful when one is in position to produce an explicit description of $(\,\mathbb{W}-z\mathbb{I}\,)^{-1}$ in which the eigenvalues stand nakedly exposed, and can be further refined when one possesses also explicit descriptions of the associated eigenvectors.

Passage to the continuous limit by "refinement of the lattice." Suppose we had physical interest in the propagation of weak compressional waves along a wire. It is known that wires are composed of atoms, and plausible that the wire might successfully be modeled by a lattice of the design considered in the preceeding section. But it seems extravagant to invoke "atomicity" in the description of a system which to eye and instrument appears to be so continuous. We are motivated to seek a dynamical formalism which conforms more naturally to the macroscopic physics of the wire-as-we-perceive-it—a *field theory* of wires which operates in the *smooth approximation*. Such a theory (which has formal/practical interest vastly deeper, it goes without saying, than the "physics of wires") can be constructed as follows:

We embed the physical lattice within a population of similarly-structured but merely "mental" lattices. All members of the population have the same total length ℓ and the same total mass M; the number N of constituent "atoms" is, however, considered to increase without bound, with the result that both the lattice constant a and the atomic mass m drop asymptotically (i.e., in the "continuous limit") to zero. The scheme, which I call "refinement of the lattice,"

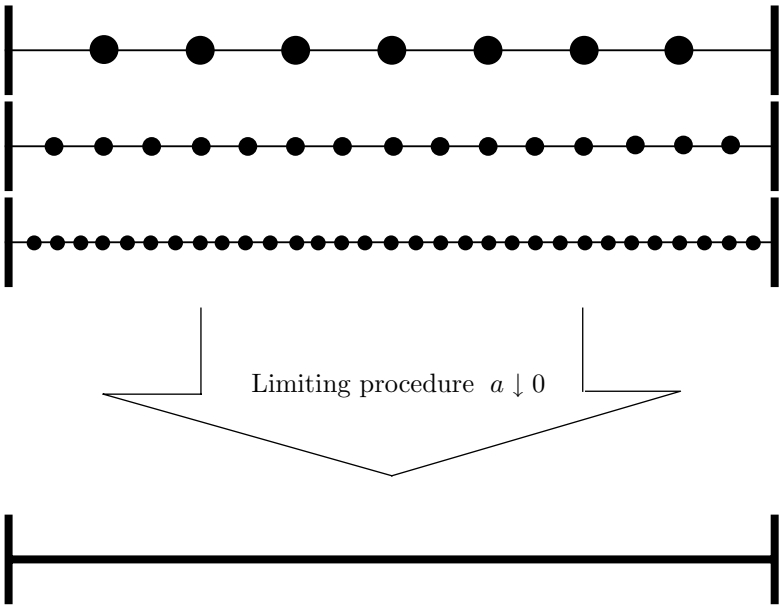


FIGURE 3: *Simple essentials of the "lattice refinement" procedure.*

is illustrated in the preceding figure. Taking the lattice constant to be our control parameter, we have

$$\left.\begin{aligned} N(a) = \frac{\ell}{a} - 1 = \frac{\ell}{a}\left(1 - \frac{a}{\ell}\right) \quad &\longrightarrow \quad \frac{\ell}{a} \\ m(a) = \frac{M}{N(a)} = \frac{M}{\ell} a \left(1 - \frac{a}{\ell}\right)^{-1} \quad &\longrightarrow \quad \mu a \end{aligned}\right\} \quad \text{for } a \ll \ell$$

where $\mu = M/\ell$ defines the *linear mass density* of the system.

As $a \downarrow 0$ the position $x_n = na$ of the n^{th} atom squeezes (for all n) ever closer to the left end of the lattice. Evidently our former practice—the ordinal enumeration of the constituent elements of our N-particle system—must be abandoned in the continuous limit. To circumvent this problem, we agree to write $\varphi(x)$ to describe the displacement (from equilibrium) of the mass element which at equilibrium resides at x $(0 \leq x \leq \ell)$. Where formerly we wrote φ_n to describe the displacement of the n^{th} atom we would now write $\varphi(x_n)$; evidently it is still possible but now no longer essential that x range on a discrete set.

In this modified notation we might, by (4), write

$$\ddot{\varphi}(x) = \frac{k}{m(a)}\big\{\varphi(x+a) - 2\varphi(x) + \varphi(x-a)\big\} \quad : \quad x = x_2, x_3, \ldots, x_{N-1}$$

to describe the motion of the typical (i.e., non-terminal) element of a discrete lattice (or "crystal"). But this equation is beautifully adapted to the formal needs of our projected "passage to the continuous limit." For we can write

$$\ddot{\varphi}(x) = \frac{k}{m(a)}a^2 \cdot \underbrace{\left\{ \frac{\frac{\varphi(x+a)-\varphi(x)}{a} - \frac{\varphi(x)-\varphi(x-a)}{a}}{a} \right\}}_{\text{becomes } \frac{\partial^2\varphi}{\partial x^2} \text{ as } a \downarrow 0}$$

and to achieve a sensible result have only to require that

$$\lim_{a\downarrow 0} \frac{ka^2}{m(a)} \sim \lim_{a\downarrow 0} \frac{ka}{\mu} = \text{a non-zero constant, call it } c^2$$

where c has necessarily the dimensionality of a "velocity" (but is not, at this point, to be associated with the "speed of light"). Evidently "regraduation of the spring constant" is a forced attribute of the lattice refinement procedure; we must have

$$k(a) = \frac{c^2\mu}{a}\left(1 + \text{inconsequential terms of order } \frac{a}{\ell}\right) \tag{7}$$

according to which the inter-atomic springs become necessarily stronger and stronger as the lattice refinement process proceeds: $k(a) \longrightarrow \infty$ as $a \downarrow 0$. This surprising development can be made intuitively intelligible by the following line of argument: Springs compose by the "law of capacitors;" for springs k_1 and k_2 connected in series one has

$$\frac{1}{k_{\text{effective}}} = \frac{1}{k_1} + \frac{1}{k_2}$$

so if one considers a spring K of length ℓ to have been assembled by connecting in series $N+1$ identical springs $k(a)$ of length $a = \ell/(N+1)$ one has

$$\frac{1}{K} = N\frac{1}{k(a)} \quad \text{with} \quad N = \frac{\ell}{a}\left(1 - \frac{a}{\ell}\right)$$

giving

$$k(a) = NK = \frac{K\ell}{a}\left(1 - \frac{a}{\ell}\right)$$

To recover (7) we have only to set $K = c^2\mu/\ell$, which we imagine to be a *constant of the refinement process*—the same in the limiting case of a compressional wire (or "string") as for the discrete crystal from which we started. Since ℓ and

μ have already been assumed to be constants of the refinement process, this amounts to a stipulation that c^2 be such a constant.

Thus is the large system (4) of coupled *ordinary* differential equations of motion seen "in the continuous limit" to go over into a *single partial* differential equation of motion:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \tag{8}$$

This, of course, is precisely the familiar "wave equation," of which the following are frequently-used notational variants:

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right\}\varphi(x,t) = 0$$

$$\left\{\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2\right\}\varphi = 0$$

$$\left\{\tfrac{1}{c^2}\partial_t^2 - \partial_x^2\right\}\varphi = 0$$

$$\tfrac{1}{c^2}\varphi_{tt} - \varphi_{xx} = 0$$

We will encounter wave equations of many types and structures before we are done, but when one speaks of *the* wave equation one invariably has in mind either (8) or its higher-dimensional generalization

$$\left\{\tfrac{1}{c^2}\partial_t^2 - \nabla^2\right\}\varphi = 0$$

The wave equation has been demonstrated to arise in what might be called the "continuous approximation" from the theory of simple crystals, but it plays a fundamental role also in contexts—electrodynamics, for example—where no "underlying atomicity" is, so far as we are aware, present in the physics. We have learned to read in the $\partial_t^2\varphi$-term an echo of the fact that, according to Newton, *acceleration* is the kinematic variable under the direct control of F/m, and to read in the $\partial_x^2\varphi$-term an allusion to the fact harmonic *nearest-neighbor* interactions dominate the physics of crystals.[2] It is tempting to suppose that the essentials of this insight pertain to *all* natural occurances of the wave equation, and it becomes interesting in this light to recall that it is ultimately from the wave equation that we acquire an interest in the Lorentz group—an interest, that is to say, in special relativity.

Wave functions, and their relation to solutions of the lattice equations. When Newton looked to the physics of one-dimensional crystals to model the acoustic

[2] Interaction with *next* nearest neighbors would introduce ∂_x^4-terms into the associated wave equation. Generally, increased non-locality entails radically increased complication of the associated field theory. For an account of the details see C. Barnes, "The dynamics of flylines and other classical strings" (Reed College, 1992), which won for its author the APS's Apker Award.

vibrations of an air column, it was because he lacked access to a well-developed theory of partial differential equations; he found it easier to contemplate the implications of (4) than to write and study (8). We, however, are in the reverse situation. Though to do so has somewhat the nature of a digression, I look now to some of the most elementary implications of (8). I proceed in special reference to this question: To what extent do solutions of the wave equation (8) serve to clarify—and to what extent to misrepresent—the physics of real crystals ($a \neq 0$)?

Let us agree henceforth to use the term "wave function" to denote any solution of the wave equation (8), which we notate $\Box\varphi = 0$ with

$$\Box \equiv \tfrac{1}{c^2}\partial_t^2 - \partial_x^2 = \left(\tfrac{1}{c}\partial_t + \partial_x\right)\left(\tfrac{1}{c}\partial_t - \partial_x\right) \tag{9}$$

Clearly

$$\begin{aligned} \left(\tfrac{1}{c}\partial_t + \partial_x\right) f = 0 \quad &\longleftrightarrow \quad f = f(x - ct) \\ \left(\tfrac{1}{c}\partial_t - \partial_x\right) g = 0 \quad &\longleftrightarrow \quad g = g(x + ct) \end{aligned}$$

where $f(\cdot)$ and $g(\cdot)$ are any differentiable functions of a single variable. It is therefore plausible (also true!) that the most general wave function can be described

$$\begin{aligned} \varphi(x,t) &= f(x - ct) + g(x + ct) \\ &= \text{ right-running waveform} + \text{ left-running waveform} \end{aligned} \tag{10}$$

The representation (10) is, it should be noted, preserved under superposition. But it will, in general, not be directly evident to the casual eye of the person who is simply watching the motion of φ. When right and left-running waves collide they do so non-interactively—by simple superposition—and emerge from their encounter unscathed/unaltered. But in typical applications $f(\cdot)$ and $g(\cdot)$ will "sense each other's structure" (i.e., be structurally correlated) in forced consequence of imposed *boundary conditions*. Thus

$$\varphi(0,t) = 0 \quad \text{(all } t\text{)}$$

entails $g(x) = -f(-x)$, while the additional requirement

$$\varphi(\ell,t) = 0 \quad \text{(all } t\text{)}$$

would force $f(\cdot)$ to be periodic: $f(x) = f(x + 2\ell)$ for all x.

Implicit already in some preceeding remarks is the important fact that from the linearity of the wave equation it follows that wave functions are subject to a *principle of superposition*:

$$\text{wave function} + \text{wave function} = \text{wave function}$$

We stand therefore in position to consider representations of the form

$$\text{complicated wave function} = \sum \text{simple wave functions}$$

as Fourier (who in point of historical fact worked in—among others—precisely this physical context) was among the first to appreciate. Following now in Fourier's footsteps, we find it natural to write

$$f(x) = \int_{\infty}^{\infty} F(k)e^{ikx}dk \quad \text{and} \quad g(x) = \int_{\infty}^{\infty} G(k)e^{ikx}dk$$

giving

$$f(x-ct) = \int_{\infty}^{\infty} F(k)e^{ik(x-ct)}dk \tag{11.1}$$
$$= F(k)\text{-weighted superposition of right-running } \textit{harmonic waves}$$
$$g(x-ct) = \int_{\infty}^{\infty} G(k)e^{ik(x+ct)}dk \tag{11.2}$$
$$= G(k)\text{-weighted superposition of left-running } \textit{harmonic waves}$$

The harmonic waves encountered above are "simple wave functions" in the sense that they spring to our attention when we define

$$\text{phase} = kx - \omega t$$

and ask: Under what condition is $e^{i\,(\text{phase})}$ a wave function? Immediately $\omega^2 - c^2k^2 = 0$, which entails

$$\omega(k) = \pm ck$$

From

$$\tfrac{d}{dt}(\text{phase}) = k\dot{x} - \omega = 0$$

we obtain

$$\dot{x} = \text{phase velocity} = \omega/k = \pm c$$

The "rigidity" of $f(x-ct)$ is traced thus to the k-independence of the phase velocities of the "harmonic wave functions" from which, according to (11.1), it can be considered to have been assembled; the Fourier components of $f(x-ct)$ move in synchrony, and the wave is said therefore to be "non-dispersive."

Impose now the boundary conditions $\varphi(0,t) = \varphi(\ell,t) = 0$ (all t) natural to the physics of our original crystal. Spatial periodicity is then, as we have seen, enforced, and we are led at length to wave functions of the form

$$\varphi(x,t) = \sum_{n=1}^{\infty} A_n \sin k_n x \cdot e^{i\omega_n t} \tag{12}$$
$$= \text{weighted superposition of harmonic } \textit{standing waves}$$

where

$$k_n = n\pi/\ell \quad \text{and} \quad \omega_n = ck_n$$
$$= n\omega_0 \quad \text{with} \quad \omega_0 = \pi c/\ell \quad : \quad n = 1, 2, \ldots$$

We have recovered the musical physics of a clamped string—familiar to every first-year student as it was familiar in its essentials already to Pythagoras by about 500 B.C. Several points, however, deserve comment:

Pretty clearly, the "harmonic standing waves" described above are the direct continuous-limit analogs of the harmonic "normal modes" of a crystal, and they are orthonormal in this analog

$$(2/\ell)\int_0^\ell \sin k_m x \cdot \sin k_n x dx = \delta_{mn}$$

of the statement $\boldsymbol{A}_m \cdot \boldsymbol{A}_n = \delta_{mn}$. But while the vibrational frequencies natural to a string are easy to describe and infinite in number

$$\omega_n = n\omega_0 \quad \text{with} \quad n = 1, 2, \ldots$$

the frequencies natural to a crystal are difficult to describe (zeros of a high order polynomial) and finite in number. The spatial form of a standing wave is similarly easy to describe: it is sinusoidal, with

$$\text{internodal distance} = \tfrac{1}{2}\,\text{wavelength} = \tfrac{1}{n}\ell$$

while the shape of a crystaline normal mode (eigenvector of a large matrix) is relatively difficult to describe. When—reversing Newton's procedure—one looks to the physics of strings to gain insight into the physics of crystals, one gains ease of analysis, but confronts this question: To what extent does the physics of strings speak reliably—and to what extent does it misrepresent—the physics of crystals?

For the same reason that one can draw sine waves on a screen only if the wavelength significantly exceeds the pixel size, we expect a crystal to be capable of supporting only those wave forms for which

$$\text{internodal distance} \gg \text{lattice constant}$$

Since for an N-atom crystal of length ℓ the lattice constant $a = \ell/(N+1)$, we have

$$\text{internodal distance} \begin{cases} > a & \text{for} \quad n = 1, 2, \ldots, N \\ = a & \text{for} \quad n = N \\ < a & \text{for} \quad n = N+1, N+2, \ldots \end{cases}$$

and find it natural to associate only the leading N string modes—those with frequencies

$$\omega \leq \omega_{\text{cutoff}} = N\omega_0$$

—with the modes of a crystal, and to dismiss the higher-frequency modes as artifacts. How accurate is that association? Intuitively we expect it to be most reliable—both spatially and temporally—when $\omega \ll \omega_{\text{cutoff}}$, and to become increasingly deceptive as $\omega \uparrow \omega_{\text{cutoff}}$. A proper answer requires, however, that we do precisely what we have been at pains thus far to avoid—that we actually

carry to completion the program sketched on page 4. This can, in fact, be done,[3] and yields

$$\omega_n^{\text{exact}} = \omega_0 \cdot \frac{2}{\pi}(N+1)\sin\left[\frac{n}{N+1}\frac{\pi}{2}\right] \quad : \quad N = 1, 2, \ldots, N$$

Evidently $\omega_n(\text{crystal}) < \omega_n(\text{associated string})$ in all cases, and

$$\omega_n(\text{crystal}) \sim \omega_n(\text{associated string}) \quad \text{only for } n \ll N$$

as illustrated in the accompanying figure. Pretty evidently, wave motion on a

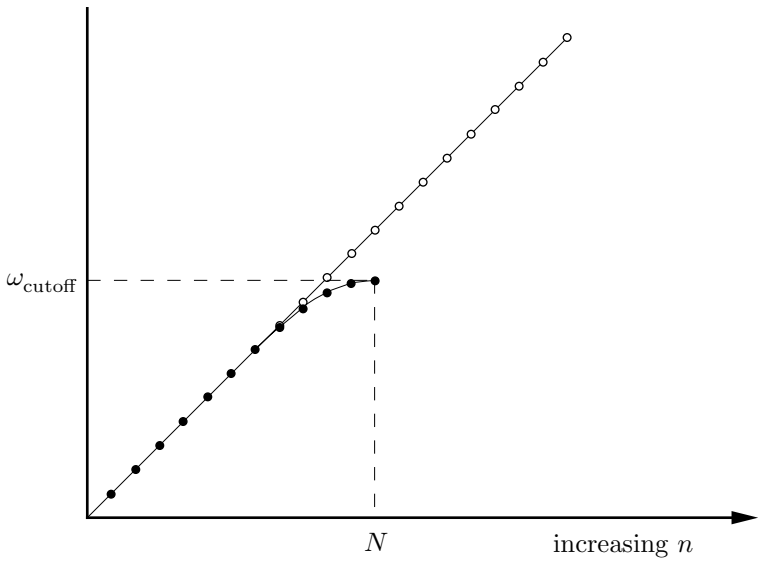


FIGURE 4: *Natural frequencies of a crystal compared with those of the associated clamped string.*

crystal is *dispersive*, and becomes (as on a string) non-dispersive only in the low-frequency limit.

If our interest attached actually to the physics of discrete N-body systems, then the moral implicit in preceeding remarks would be clear: *field-theoretic*

[3] See pp. 67–68 of U. Grenander & G. Szegö, *Toeplitz Forms and Their Applications* (1958), who exploit the fact that the $\mathbb{S}$-matrix in (5) is of such specialized structure as to comprise an instance of a "Toeplitz matrix," about which much is known. Related material can also be found in §3 of E. Montroll, "Markoff Chains, Wiener Integrals, and Quantum Theory," Comm. Pure & Appl. Math. **5**, 415 (1952). For more immediately physical discussion see W. Thompson, *Theory of Vibrations with Applications* (1972), which is the source of the result quoted on p. 50 of R. Blevins, *Formulas for Natural Frequency and Mode Shape* (1979) and reproduced here.

methods may be latently a source of striking analytic power, but must be used with cautious circumspection. In fact our interest attaches primarily to those "field-theoretic methods" themselves. For us, crystals are mere workshops, of interest primarily for such clues as they may provide concerning how we, as field-theorists, should be conducting our affairs. And so they will function in the discussion to which we now turn.

Lagrangian formulation of the wave equation. The equations of motion (4) of our one-dimensional crystal were at (3) obtained from a Lagrangian

$$L = \sum_1^N \tfrac{1}{2}m\dot{\varphi}_n^2 - \tfrac{1}{2}k\varphi_1^2 - \sum_1^{N-1} \tfrac{1}{2}k(\varphi_{n+1} - \varphi_n)^2 - \tfrac{1}{2}k\varphi_N^2 \tag{13}$$

which in notation adapted to the realities of the a-parameterized refinement process reads

$$L = \sum_1^N \tfrac{1}{2}\mu a\dot{\varphi}^2(x_n) - \tfrac{1}{2}\frac{c^2\mu}{a}\varphi^2(a) - \sum_1^{N-1} \tfrac{1}{2}\frac{c^2\mu}{a}\left[\varphi(x_n + a) - \varphi(x_n)\right]^2 - \tfrac{1}{2}\frac{c^2\mu}{a}\varphi^2(\ell - a)$$

or again (which is for our purposes more useful)

$$L = \sum_1^N \tfrac{1}{2}\mu a\dot{\varphi}^2(x_n) - \sum_1^{N-1} \tfrac{1}{2}\mu c^2 a\left[\frac{\varphi(x_n + a) - \varphi(x_n)}{a}\right]^2 - \tfrac{1}{2}\mu c^2 \cdot \tfrac{1}{a}\varphi^2(a) - \tfrac{1}{2}\mu c^2 \cdot \tfrac{1}{a}\varphi^2(\ell - a)$$

In the continuous limit $a \downarrow 0$ the dangling terms vanish in consequence of the conditions $\varphi(0) = \varphi(\ell) = 0$, and the sums become integrals; we obtain

$$L = \int_0^\ell \tfrac{1}{2}\mu c^2 \left\{ \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 \right\} dx$$

It becomes natural in this light to write

$$L = \int_0^\ell \mathcal{L}dx \tag{14}$$

with

$$\mathcal{L} = \tfrac{1}{2}\mu c^2 \left\{ \tfrac{1}{c^2}\varphi_t^2 - \varphi_x^2 \right\} \tag{15}$$

and to call $\mathcal{L}$ the "Lagrangian *density*." Since μ signifies mass density, μc^2 has the physical dimensionality of an energy density, and we have

$$[\mathcal{L}] = \text{energy/length} = \text{energy density}$$

We stand now in position (i) to trace the crystaline Lagrangian (13) to its continuous limit (14), and (ii) to trace the associated system (4) of coupled

equations of motion to its continuous limit (8). It becomes natural at this point to ask: Can the field equation (8) be obtained *directly* from the Lagrangian density (15)? A little experimentation leads to the observation that

$$\tfrac{1}{c^2}\varphi_{tt} - \varphi_{xx} = 0 \quad \text{can be formulated} \quad \left\{ \frac{\partial}{\partial t}\frac{\partial}{\partial \varphi_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \varphi_x} \right\} \mathfrak{L} = 0$$

Moreover, we could in fact (since the Lagrange density of (15) displays no *explicit* φ-dependence) write

$$\left\{ \underbrace{\frac{\partial}{\partial t}\frac{\partial}{\partial \varphi_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \varphi_x}}_{\text{one such term for each independent variable}} - \frac{\partial}{\partial \varphi} \right\} \mathfrak{L} = 0 \tag{16}$$

if we wanted to maximize formal resemblance to the Lagrange equation

$$\left\{ \frac{d}{dt}\frac{\partial}{\partial \dot{\varphi}} - \frac{\partial}{\partial \varphi} \right\} L = 0$$

familiar from particle mechanics (in which context t is the *solitary* independent variable).

It is interesting to note that by nothing more complicated than a sign change ($L = T - U \longrightarrow E = T + U$) we obtain this description

$$E = \int_0^\ell \mathcal{E}\, dx \quad \text{with} \quad \mathcal{E} = \tfrac{1}{2}\mu c^2 \left\{ \tfrac{1}{c^2}\varphi_t^2 + \varphi_x^2 \right\} \tag{17}$$

of the *energy* resident on our vibrating string. That (global) energy conservation

$$\dot{E} = 0$$

is an implication of the field equation (i.e., of the equation of motion) can be established as follows:

$$\dot{E} = \mu c^2 \int_0^\ell \left\{ \tfrac{1}{c^2}\varphi_t \varphi_{tt} + \varphi_x \varphi_{xt} \right\} dx$$

But by assumption φ satisfies $\frac{1}{c^2}\varphi_{tt} - \varphi_{xx} = 0$, so

$$\begin{aligned}
&= \mu c^2 \int_0^\ell \left\{ \varphi_t \varphi_{xx} + \varphi_x \varphi_{xt} \right\} dx \\
&= \mu c^2 \int_0^\ell \frac{\partial}{\partial x}\left(\varphi_t \varphi_x \right) dx \\
&= \mu c^2 \left. \varphi_t \varphi_x \right|_0^\ell \\
&= 0 \quad \text{since } \varphi(0,t) = \varphi(\ell,t) = 0 \text{ (all } t\text{) entails } \varphi_t(0,t) = \varphi_t(\ell,t) = 0
\end{aligned}$$

Later we will be in position to discuss the deeper origins of a large population of statements of which energy conservation is but an illustrative example.

Field-theoretic formulation of Hamilton's principle. In the classical mechanics of particles it is possible to dismiss the statement

$$\delta \int L dt = 0 \quad \Longleftrightarrow \quad \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} \right\} L = 0 \quad : \quad i = 1, 2, \ldots, n$$

as but an elegant curiosity, for one enters into discussion of Hamilton's principle already in full command—thanks *via* Lagrange to Newton—of the equations of motion. The history of field theory supplies, however, no "Newton"—no ready-made general formulation of the equations of motion. In field theory it is, as will emerge, a generalization of Hamilton's principle which steps into the breech. The elegantly simple statement $\delta S = 0$ acquires in field theory a force and a degree of practical utility far beyond anything for which our pre-field-theoretic experience has prepared us. It becomes the central unifying principle of our subject—its workhorse.

Our objective here will be to establish the sense in which

$$\delta S = 0 \quad \Longrightarrow \quad \text{field equations}$$

but before we play chess we must put the pieces on the board. We begin by noticing that while a single field $\varphi(x,t)$ served to describe the longitudinal vibration of a clamped string, two fields—call them $\varphi_1(x,t)$ and $\varphi_2(x,t)$—would be required to describe the *transverse* vibration of such a system. And if our string had non-vanishing cross section we might find it necessary[4] to introduce yet another field $\varphi_3(x,t)$ to describe its torsional motion. Evidently a (finite) *set* of field functions

$$\varphi_1(x,t),\ \varphi_2(x,t),\ \ldots,\ \varphi_N(x,t)$$

will in the general case be required to describe the state of a distributed system, and these will, in the general case, be *dimensionally diverse*.[5]

Our string field φ was a t-dependent structure defined on a line, but in general our field systems $\varphi_1, \varphi_2, \ldots, \varphi_N$ (collectively denoted φ) will reside on manifolds of *several* dimensions. We write $x^1, x^2, \ldots, x^n$ (collectively $\boldsymbol{x}$) to refer to some specified coordinatization of the manifold. Typically we will have $n = 3$ and x^1, x^2, x^3 will refer to a *Cartesian* coordinate system, but by no means—consider the field that lives on a torus—will that be universally the case.

[4] See in this connection §6.1 "Generation of torsional waves by bow-friction forces" in L. Cremer, *The Physics of the Violin* (1981).

[5] In the preceeding example φ, φ_1 and φ_2 refer to spatial displacements, and have therefore the dimensionality of length, while the angular variable φ_3 is dimensionless.

Only rarely in particle mechanics do the physical constants which enter into the description of a system permit the formation of a "natural velocity." In field theory, on the other hand, it is rare when the available physical constants do *not* permit the formation of one or more natural velocities—constants $c_1, c_2, \ldots$ which describe (or at any rate enter into the description of) the "rapidity with which effects propagate."[6] In this respect the "velocity of light" $c = 1/\sqrt{\epsilon_0 \mu_0}$ is entirely typical, though it is atypical in that it is assigned by special relativity a preferred role which in the end has nothing special to do with light! In the presence of such a constant it becomes possible to write $x^0 = ct$, and natural in place of $\varphi(t, \boldsymbol{x})$ to adopt the still more compact notation $\varphi(x)$. This we will frequently find it convenient to do, and collaterally to write simply $\partial\varphi$ when we have in mind the entire population of first partials

$$\varphi_{\alpha,i} = \frac{\partial \varphi_\alpha}{\partial x^i} \quad \text{where} \quad \begin{cases} \alpha = 1, 2, \ldots, N \\ i \ = 0, 1, \ldots, n \end{cases}$$

At a deeper level, these opportunistic adjustments invite one to think of the fields $\varphi(x)$ not as objects that move on an n-dimensional manifold, but as objects that inhabit an $(n+1)$-dimensional spacetime.

Our chess board is now set up; it is time to play the game. As an opening move, we assume a Lagrange density of the form $\mathcal{L}(\varphi, \partial\varphi, x)$ to have been given.[7] Within the particular context provided by our clamped string system we find it natural to introduce an *action functional* by writing

$$S[\varphi(x,t)] = \int_{t_1}^{t_2} L dt \quad \text{with} \quad L = \int_0^\ell \mathcal{L}(\varphi, \partial\varphi, x) dx$$

or, more compactly,

$$S[\varphi(x,t)] = \iint_{\mathcal{R}} \mathcal{L}(\varphi, \partial\varphi, x)\, dt dx$$

where $\mathcal{R}$ refers to the "rectangular box" in spacetime defined $t_1 \leq t \leq t_2$, $0 \leq x \leq \ell$. By straightforward extension, we take $\mathcal{R}$ to be an *arbitrary* domain (or "bubble") in $n+1$-dimensional spacetime and agree to let

$$S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \mathcal{L}(\varphi, \partial\varphi, x) dt dx^1 \cdots dx^n \tag{18}$$

serve in the general case to define the "action functional relative to $\mathcal{R}$" of the field system $\mathcal{L}(\varphi, \partial\varphi, x)$: $\varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\}$. In the following figure I have

[6] The non-relativistic quantum mechanics of a particle—thought of as a classical field theory—is in this respect the great exception.

[7] Note the assumed absence of arguments of the type $\partial\partial\varphi, \partial\partial\partial\varphi, \ldots$ Such terms, were we proceeding from a crystaline model, would reflect the presence of next-nearest-neighbor and even more remote interactions. Evidently we proceed subject to a tacit *locality* assumption.

attempted to represent the geometrical image one has in mind when one draws upon the fundamental definition (18).

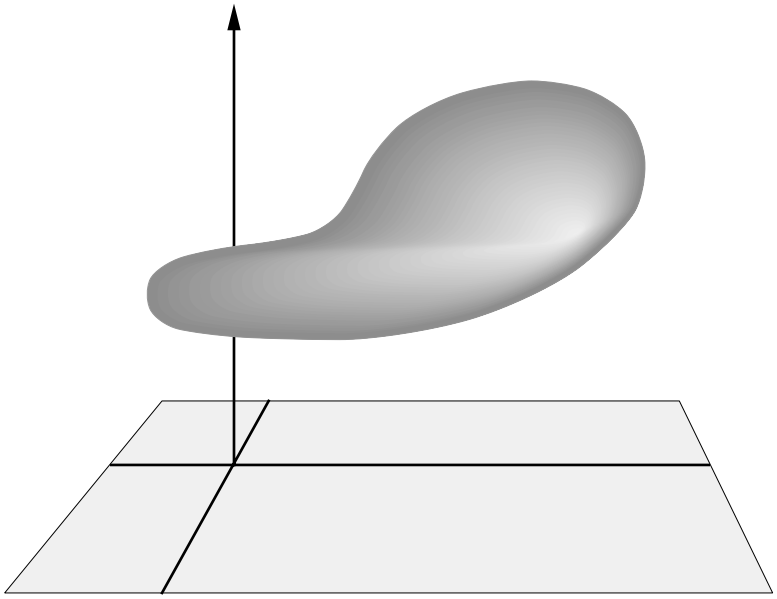


FIGURE 5: *A "bubble" $\mathcal{R}$ in the $(n+1)$-dimensional spacetime inhabited by the field system $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$*

We look to the leading-order response $S_{\mathcal{R}} \longrightarrow S_{\mathcal{R}} + \delta S_{\mathcal{R}}$ of the action functional $S_{\mathcal{R}}[\varphi]$ to hypothetical variation $\varphi \longrightarrow \varphi + \delta\varphi$ of the field system, subject to the explicit stipulation that

$$\delta\varphi = 0 \quad \text{on the boundary } \partial\mathcal{R} \text{ of } \mathcal{R}$$

Writing[8]

$$\begin{aligned}\delta S_{\mathcal{R}}[\varphi] &= S_{\mathcal{R}}[\varphi + \delta\varphi] - S_{\mathcal{R}}[\varphi] \\ &= \int_{\mathcal{R}} \big\{\mathcal{L}(\varphi + \delta\varphi, \partial\varphi + \delta\partial\varphi, x) - \mathcal{L}(\varphi, \partial\varphi, x)\big\} dx\end{aligned}$$

[8] I adopt here and henceforth the abbreviation

$$dx = dt dx^1 \cdots dx^n = \tfrac{1}{c} dx^0 dx^1 \cdots dx^n$$

where c has been selected from the population of velocities $c_1, c_2, \ldots$ natural to the system in hand. The latter variant, though almost always available in principle, will acquire special naturalness and utility in connection with the theory of *relativistic* classical fields.

we (by Taylor expansion of the integrand) obtain

$$\delta S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} \delta\varphi_\alpha + \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \delta\varphi_{\alpha,i} \right\} dx$$

where $\sum_\alpha$ and $\sum_i$ are understood. Pretty clearly

$$\delta\varphi_{\alpha,i} = \partial_i \left(\delta\varphi_\alpha\right)$$

so, integrating by parts, we have

$$\delta S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \delta\varphi_\alpha \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \right\} dx + \int_{\mathcal{R}} \frac{\partial}{\partial x^i} \left(\delta\varphi_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \right) dx$$

But the second of the integrals on the right has the structure $\int_{\mathcal{R}} (\partial_i \mathcal{A}^i) dx$, and by the divergence theorem[9] $\int_{\mathcal{R}} (\partial_i \mathcal{A}^i) dx = \int_{\partial\mathcal{R}} \mathcal{A}^i d\sigma_i$ so

$$\int_{\mathcal{R}} \frac{\partial}{\partial x^i} \left(\delta\varphi_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \right) dx = \int_{\partial\mathcal{R}} \left(\delta\varphi_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \right) d\sigma_i$$

which vanishes since, by assumption, $\delta\varphi = 0$ on the boundary $\delta\mathcal{R}$ of $\mathcal{R}$. Therefore

$$\delta S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \delta\varphi_\alpha \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \right\} dx \tag{19}$$

By *Hamilton's Principle*, φ will be "dynamical" if and only if

$$\delta S_{\mathcal{R}}[\varphi] = 0 \quad \text{for all regions } \mathcal{R} \text{ and all variations } \delta\varphi \tag{20}$$

It follows from (19) that if the field system φ conforms to Hamilton's Principle then the field functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ have necessarily to be solutions of the *field equations*

$$\left\{ \frac{\partial}{\partial \varphi_\alpha} - \frac{\partial}{\partial x^i} \frac{\partial}{\partial \varphi_{\alpha,i}} \right\} \mathcal{L}(\varphi, \partial\varphi, x) = 0 \qquad \alpha = 1, 2, \ldots, N \tag{21}$$

These comprise an N-fold system of coupled second-order partial differential equations. Equations (21) are of a form which was anticipated already at (16), and can in a more explicit notation be expressed

$$\frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \frac{\partial^2 \mathcal{L}}{\partial \varphi_\beta \partial \varphi_{\alpha,i}} \varphi_{\beta,i} - \frac{\partial^2 \mathcal{L}}{\partial \varphi_{\beta,j} \partial \varphi_{\alpha,i}} \varphi_{\beta,ij} - \boldsymbol{\partial}_i \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} = 0 \tag{22}$$

where $\sum_\alpha$, $\sum_i$ and $\sum_j$ are understood, and where $\boldsymbol{\partial}_i$ looks only to the *explicit* x-dependence of $\mathcal{L}(\cdot, \cdot, x)$. The field equations will, in the general case, be non-linear.

[9] Which is to say, by Gauß' theorem—by a particular instance of Stokes' theorem. Here $d\sigma_i$ signifies an outer-directed surface element.

Gauge freedom in the construction of the Lagrangian. The Lagrangian density $\mathcal{L}$ plays in field theory precisely the "system characterizer" role which in particle mechanics is played by the Lagrangian, $L(q, \dot{q}, t)$. The

$$\text{field system} \quad \longleftrightarrow \quad \text{Lagrangian density}$$

association is, however—like its particle mechanical counterpart—non-unique. If $\mathcal{L}$ and $\mathcal{L}'$ stand in the relation

$$\mathcal{L}'(\varphi, \partial\varphi, x) = \mathcal{L}(\varphi, \partial\varphi, x) + \partial_k \mathcal{G}^k(\varphi, x)$$

then they *give rise to identical field equations*, for the simple reason that (as can be shown by explicit calculation)

$$\left\{ \frac{\partial}{\partial \varphi_\alpha} - \frac{\partial}{\partial x^i} \frac{\partial}{\partial \varphi_{\alpha,i}} \right\} \partial_k \mathcal{G}^k(\varphi, x) = 0 \quad \text{identically, for all } \mathcal{G}^k(\varphi, x)$$

Insight into the origin of this important fact follows from the observation that the gauge transformation

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \partial_k \mathcal{G}^k \tag{23}$$

induces

$$\begin{aligned} S_{\mathcal{R}} = \int_{\mathcal{R}} \mathcal{L} dx \quad \longrightarrow \quad S_{\mathcal{R}}{}' &= \int_{\mathcal{R}} \mathcal{L}' dx \\ &= S_{\mathcal{R}} + \int_{\mathcal{R}} \partial_k \mathcal{G}^k dx \\ &= S_{\mathcal{R}} + \int_{\partial\mathcal{R}} \mathcal{G}^k d\sigma_k \quad \text{by the divergence theorem} \end{aligned}$$

and that the final term—the boundary term—is, for the purposes of Hamilton's Principle, invisible.

Non-uniqueness—gauge freedom—entails that the Lagrangian density $\mathcal{L}$ is, its formal importance notwithstanding, not itself directly "physical." It is, in this respect, reminiscent of the potentials $U(x)$ of particle mechanics, which are determined only up to gauge transformations of the form

$$U(x) \longrightarrow U'(x) = U(x) + \text{constant}$$

In the laboratory one measures not potentials, but only such gauge-invariant constructs as (for example) potential *differences*. Gauge freedom imposes upon us an obligation frequently to attach (at least tacitly) tedious qualifications to statements we might prefer to keep sharply simple. For example: if $\mathcal{L}$ depends *quadratically* upon φ and $\partial\varphi$ then (see again (22)) the associated field equations will be *linear*. It would, however, be incorrect to state that "linearity implies

the quadraticity" of $\mathcal{L}$, for to $\mathcal{L}$ we could aways add a quadraticity-breaking gauge term.

We have now in hand, in the field equations (21), the general field-theoretic proposition of which (16) provided our first hint. We turn now to a discussion the objective of which will be to establish in similar generality the origins of a population of important propostions of which (17) is the precursor. We turn, in short, to a discussion of "Hamilton's Principle, Part II"—i.e., of Noether's Theorem in its original (classical field-theoretic) setting.

Field-theoretic formulation of Noether's Theorem. Emmy Noether (1882–1935) is today remembered by mathematicians primarily for the importance of her contributions to algebraic number theory, especially to the theory of ideals and to several aspects of the theory of invariants. But her name will live forever among physicists for the work which is our present subject matter. I digress to sketch the soil from which that work sprang.

Noether did the first semester of what we would today call graduate study at the University of Göttingen (winter term 1903–1904), where she audited[10] lectures by (among others) Hermann Minkowski, Felix Klein and David Hilbert. She then returned to Erlangen, where her father was a professor, where Felix Klein (1849–1925) had in his inaugural lecture (1872) propounded the influential "Erlangen Program"[11] which held the *group* and *invariance* concepts to be among the central organizing principles of mathematical (and also physical) thought, and where Paul Gordon (1837–1912) became her mentor. Research publications by E. Noether began appearing in 1907. Meanwhile...

Einstein was at the sublime height of his powers during the "miracle decade" 1905–1915, and his work—especially that relating to the development of general relativty—attracted the close attention of mathematicians, especially (at Göttingen) of Klein and Hilbert. In 1915 Noether—though unable because of her sex to obtain either an advanced degree or a paid teaching position—returned to Göttingen, where she soon became a kind of unofficial assistant to Klein and especially Hilbert. In November of 1915 Noether wrote to a friend back in Erlangen that "the theory of invariants is the thing here now; even the physicist Hertz[12] is studying [the subject]; Hilbert plans to lecture next week about his ideas on Einstein's differential invariants, and so our crowd had better be ready." To another friend she reported that she and collaborators were carrying out calculations of the most difficult kind for Einstein "although none of us understands what they are for." Klein remarks in a letter to Hilbert

[10] Women were still at the time permitted to audit, but not to enroll as students in advanced courses of study.

[11] For a good discussion of the historical impact of the Erlangen Program see E. Bell, *Development of Mathematics* (1945), pp. 442–453.

[12] The reference is to Gustav Hertz (1887–1975), later to acquire fame for his participation in the celebrated "Franck–Hertz experiment." The theoretically astute Heinrich Hertz certainly would have had interest in such material, but had died already in 1894.

that "Noether is continually advising me in my projects, and it is really through her that I have become competent in the subject..." Hilbert makes reference in his response to "Emmy Noether, whom I called upon to help me with such questions as my theorem on the conservation of energy..." It was Noether's effort to be "helpful" in precisely that connection which led to the development of "Noether's Theorem."

By 1919 it had finally become possible (owing to a change in German law; manpower had become short in a Germany at war) for a woman to earn an advanced degree, and to hold a university appointment. On 4 June 1919—six days after A. S. Eddington had obtained solar eclipse data in agreement with Einstein's prediction of the bending of star light—Noether stood before a mathematical faculty which included R. Courant, P. Debye, Hilbert, Klein, E. Landau, L. Prandtl and W. Voigt to deliver her *Habilitation* lecture. She had been active in many areas during her years at Göttingen, and could have spoken on a wide variety of topics. But she chose in fact to speak on research which she had published already in 1918 under the title "Invariante Variationsprobleme." Concerning that work she wrote at the time as follows:

> "The last...of the works to be mentioned here concern differential invariants and variational problems and in part are an outgrowth of my assistance to Klein and Hilbert in their work on Einstein's general theory of relativity...The [paper], which I designated as my *Habilitation* thesis, deals with arbitrary finite or infinite continuous groups, in the sense of Lie, and discloses what consequences it has for a variational problem to be invariant with respect to such a group. The general results contain, as special cases, the theorems on first integrals as they are known in mechanics; furthermore, the conservation theorems and the interdependences among the field equations in the theory of relativity—while, on the other hand, the converses of these theorems are also given..."

Short biography does invariable violence to the always-intricate facts of the matter. For those I must refer you, dear reader, to the relevant literature,[13] which anticipate, I think likely to make a lasting impression upon you. Here my objective has been simply to suggest that the work for which Noether's name will forever be remembered by physicists is work which is in fact clearly consonant with the principal themes evident in the larger body of her mathematical work. It very cleverly exploits and enshrines a little constellation of ideas which were very much in the air—at Göttingen and elsewhere—during

[13] My principal source has been A. Dick's *Emmy Noether* (1981), but see also *Emmy Noether: A Tribute to Her Life and Work* (edited by J. Brewer & M. Smith, and published in that same centennial year) and the deeply informed and sensitively written obituary by Hermann Weyl which can be found at p. 425 in Volume III of his *Gesammelte Adhandlugen* (1968). The circumstances associated specifically with the development of Noether's Theorem are discussed on p. 431.

the first years of this century, and which have assumed ever greater importance as the century has matured.

Noether's Theorem emerges fairly spontaneously when two ideas are (so to speak) "rubbed against each other." The first of those has to do with the concept of "dynamical action," the other with the concept of "parameterized map." We consider them in that order:

Let $\mathfrak{L}(\varphi, \partial\varphi, x)$ be given, and let $\varphi(x)$ be some *solution* of the associated field equations. We agree to write $\varphi_{\text{dynamical}}(x)$ when we wish to emphasize that it is *such* field functions—"dynamical" field functions—that we have in mind. The phrase "dynamical action" refers then to constructions of the form

$$S_{\mathcal{R}}[\varphi_{\text{dynamical}}(x)]$$

Easy enough... yet complex enough to conceal some deep mysteries, as comes quickly to light when one looks to the corresponding construct in ordinary particle mechanics. Consider $L(q, \dot{q}, t)$ to be given, and take $q_{\text{dynamical}}(t)$ to be a solution of the associated equations of motion which conforms to endpoint conditions

$$q_{\text{dynamical}}(t) = \begin{cases} q_1 & \text{when} \quad t = t_1 \\ q_2 & \text{when} \quad t = t_2 \end{cases}$$

Familiarly, the action functional $S[q(t)]$, when evaluated at $q(t) = q_{\text{dynamical}}(t)$, becomes a *function of the endpoint data*:

$$S[q_{\text{dynamical}}(t)] = S(q_2, t_2; q_1, t_1)$$

But while initial data

$$q(t_1) = q_1 \quad \text{and} \quad \dot{q}(t_1) = v_1$$

generally is sufficient to determine $q(t) = q_{\text{dynamical}}(t)$ uniquely, endpoint data generally is not; evidently the dynamical action function must, in the general case, be multi-valued. This important fact we attribute to the circumstance that while statements of the form

$$S[q(t)] = \text{extremum}$$

are global in character (and might for that reason be expected to admit in most cases of a unique solution), Hamilton's Principle $\delta S[q(t)] = 0$ imposes only a *local* condition on the trajectory $q(t)$. Returning in this light to classical field theory, we expect to have

$$S_{\mathcal{R}}[\varphi_{\text{dynamical}}(x)] = \text{some function } S(\varphi(\partial\mathcal{R})) \text{ of prescribed } \textit{boundary data}$$

Moreover, we expect $S(\varphi(\partial\mathcal{R}))$ to be in the general case *multi-valued*. But it is by no means obvious that there even *exists* a $\varphi_{\text{dynamical}}(x)$ which conforms to arbitrarily prescribed boundary data $\varphi(\partial\mathcal{R})$, and to resolve such an issue one would have to enter distractingly far into the general theory of partial

differential equations. Happily, we can proceed formally in total ignorance of such theory, and address such matters on a case by case basis as specific occasions arise.

Turning now to the concept of "parameterized map" as it enters into Noether's train of thought... the simplest manifestation of basic idea emerges naturally as soon as one agrees to look upon rotations, translations, dilations, curvilinear deformations and other such point-to-point transformations as "flows" achieved by specification of one or more continuously variable "control parameters" $\omega = \{\omega^1, \omega^2, \ldots, \omega^\nu\}$. More concretely, let a coordinate system be inscribed on the $(n+1)$-space inhabited by our field system, let x and X signify the coordinates of a point and its image, write

$$\mathcal{T}_\omega \quad : \quad x \longrightarrow X(x;\omega) \tag{24}$$

and (though it entails the exclusion of such otherwise unexceptionable—and frequently important—transformations as reflections and projections) agree to look henceforth only to cases in which the set $\mathcal{T}=\{\mathcal{T}_\omega\}$ has these properties:

- *compositional closure*: For every pair $\{\omega_1, \omega_2\}$ there exists an $\omega(\omega_1, \omega_2)$ such that $\mathcal{T}_{\omega_2}\mathcal{T}_{\omega_1} = \mathcal{T}_{\omega(\omega_1,\omega_2)}$
- *existence of an identity*: There exists within $\mathcal{T}$ an element $\mathcal{T}_{\omega_0}$ such that $X(x;\omega_0) = x$ for all x ; we henceforth assume the parameterization to have been rigged in such a way as to achieve $\omega_0 = 0$, and write $\mathcal{T}_0$ to denote the identity transformation.
- *existence of an inverse*: For every ω there exists an $\Omega(\omega)$ such that $\mathcal{T}_\Omega\mathcal{T}_\omega = \mathcal{T}_0$; in other words, $X(X(x;\omega);\Omega(\omega)) = x$ for all x and all ω.
- *associativity*: $\mathcal{T}_{\omega_3}(\mathcal{T}_{\omega_2}\mathcal{T}_{\omega_1}) = (\mathcal{T}_{\omega_3}\mathcal{T}_{\omega_2})\mathcal{T}_{\omega_1}$

We are brought thus to the notion of a "continuous group of transformations," of which $\omega(\omega_1, \omega_2)$ is, in effect, the "group multiplication table." The theory of such groups—"Lie groups," as they are called—was (together with the details of a great many illustrative applications) worked out almost single-handedly by the Norwegian mathematician Sophus Lie (1842-1899) during the 1870's and 1880's.[14] Fundamental to the theory of Lie groups is the insight that finite transformations can be built up by iteration of infinitesimal ones; the structure of a Lie group is *latent already in its structure in the infinitesimal neighborhood of the identity.* Returning in this light to (24)—which, as we have rigged things, reduce to description of the map as experienced in the

[14] Lie and the precocious Klein (seven years his junior) had been students together at Göttingen. Klein was initially Lie's collaborator, and stood always ready to lend him support and encouragement. And Klein was, as I have remarked, one of Noether's primary mentors; the ideas here at work were therefore entirely natural to her. For a good brief account of Lie's work in its original setting (unified theory of differential equations), see Chapter V of E. Ince's classic *Ordinary Differential Equations* (1926).

neighborhood of the point $\omega = 0$ in parameter space—we have

$$\mathcal{T}_{\delta\omega} \quad : \quad x \longrightarrow X(x;\delta\omega) = x + \delta_\omega x$$

with

$$\delta_\omega x = \sum_{r=1}^{\nu} \mathcal{X}_r(x)\delta\omega^r$$

where the functions $\mathcal{X}_r(x)$—called "structure functions" because it is they which account ultimately for the distinctive structure of the particular Lie group in hand—can, in the notation natural to the finite transformation (24), be described

$$\mathcal{X}_r(x) = \left.\frac{\partial X(x;\omega)}{\partial \omega^r}\right|_{w=0}$$

The (infinitesimal) "parameterized maps" (my terminology) contemplated by Noether appropriate, but at the same time enlarge upon, the root idea sketched above. The map $\mathcal{T}_{\delta\omega}$ is understood by Noether to be "bipartite," in this sense: it sends spacetime points to new spacetime points (in precisely the manner described above), and—simultaneously but quite independently—it *adjusts the functional structure* of the field functions $\varphi(x)$:

$$\mathcal{T}_{\delta\omega} \quad : \quad \begin{cases} x \longrightarrow X(x\,;\delta\omega) = x + \delta_\omega x \\ \varphi(x) \longrightarrow \Phi(X;\delta\omega) = \varphi(x) + \delta_\omega\varphi(x) \end{cases} \tag{25}$$

where (installing all indices, but surpressing a $\sum_r$)

$$\delta_\omega x^i = \mathcal{X}_r^i(x)\delta\omega^r \tag{26.1}$$

$$\delta_\omega\varphi_\alpha(x) = \Phi_{\alpha r}(x)\delta\omega^r \tag{26.2}$$

The field variation $\delta_\omega\varphi_\alpha(x)$ derives, as emphasized above, from two distinct sources, and those contributions are (since we are working in lowest order) additive; we have

$$\begin{aligned} \delta_\omega\varphi_\alpha(x) = {} & \text{contribution from variation of } \textit{argument} \\ & + \text{contribution from variation of } \textit{functional form} \end{aligned}$$

Since the former can be described $\varphi_{\alpha,i}\delta_\omega x^i$ we can notate the preceeding disentanglement as follows:

$$\begin{aligned} \delta_\omega\varphi_\alpha &= \varphi_{\alpha,i}\delta_\omega x^i + \left\{\Phi_{\alpha r}\delta\omega^r - \varphi_{\alpha,i}\delta_\omega x^i\right\} \\ &= \varphi_{\alpha,i}\delta_\omega x^i + \underbrace{\left\{\Phi_{\alpha r} - \varphi_{\alpha,i}\mathcal{X}_r^i\right\}}\delta\omega^r \\ &\qquad\qquad = \Delta_\omega\varphi_\alpha \end{aligned} \tag{27.1}$$

Similarly

$$\delta_\omega\varphi_{\alpha,i} = \varphi_{\alpha,ij}\delta_\omega x^j + \Delta_\omega\varphi_{\alpha,i} \quad \text{with} \quad \Delta_\omega\varphi_{\alpha,i} = \left(\Delta_\omega\varphi_\alpha\right)_{,i} \tag{27.2}$$

which serves to disentangle the variations of the various field derivatives.

Armed as we are with some understanding of the concepts of "dynamical action" and "parameterized map," we are in position now at last to put those ideas into the same pot and stir; we will find that Noether's Theorem emerges (as I have claimed) "fairly spontaneously," but not without the exercise of some trickery. We look to the description of

$$\delta_\omega S_{\mathcal{R}}[\varphi] = S_{\mathcal{R}+\delta\mathcal{R}}[\Phi] - S_{\mathcal{R}}[\varphi]$$

subject to the assumption that φ is a solution of the field equations. Writing

$$S_{\mathcal{R}+\delta\mathcal{R}}[\Phi] = \int_{\mathcal{R}+\delta\mathcal{R}} \mathfrak{L}(\Phi(X), \ldots) dX$$

we have

$$= \int_{\mathcal{R}} \mathfrak{L}(\Phi(X(x)), \ldots) \left| \frac{\partial X}{\partial x} \right| dx$$

after the indicated change of variables.[15] Therefore (introducing a term at the beginning only to subtract it again at the end)

$$\delta_\omega S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \left\{ \mathfrak{L}(\Phi(X(x)), \ldots) - \mathfrak{L}(\varphi(x), \ldots) \right\} dx + \int_{\mathcal{R}} \mathfrak{L}(\Phi(X(x)), \ldots) \left\{ \left| \frac{\partial X}{\partial x} \right| - 1 \right\} dx$$

Expansion of the Jacobian (use $\det(\mathbb{I} + \epsilon\mathbb{M}) = 1 + \epsilon \mathrm{tr}\mathbb{M} + \cdots$) gives

$$\left\{ \left| \frac{\partial X}{\partial x} \right| - 1 \right\} = \frac{\partial}{\partial x^k} (\delta_\omega x^k) + \cdots$$

Since this expression is itself of first order, and we are working only in first order, we can in the second integral replace $\mathfrak{L}(\Phi(X(x)), \ldots)$ by its zeroth order approximation $\mathfrak{L}(\varphi(x), \ldots)$, giving

$$\delta_\omega S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \left\{ \mathfrak{L}(\Phi(X(x)), \ldots) - \mathfrak{L}(\varphi(x), \ldots) \right\} dx + \int_{\mathcal{R}} \mathfrak{L}(\varphi(x), \ldots) \frac{\partial}{\partial x^k} (\delta_\omega x^k) dx$$

Turning our attention now to the first of the integrals in the preceeding equation, we use

$$\Phi_\alpha(X) = \varphi_\alpha(x) + \delta_\omega \varphi_\alpha(x) \quad \text{and} \quad \Phi_{\alpha,k}(X) = \varphi_{\alpha,k}(x) + \delta_\omega \varphi_{\alpha,k}(x)$$

to obtain

[15] Noting that every X in $\mathcal{R} + \delta\mathcal{R}$ is the image under $\mathcal{T}_{\delta\omega}$ of an x in $\mathcal{R}$, we have elected to adopt the latter as our variables of integration. This has the effect of making both integrals range on the same domain.

$$\int_{\mathcal{R}} \{\mathcal{L}(\Phi(X(x)),\ldots) - \mathcal{L}(\varphi(x),\ldots)\}dx$$
$$= \int_{\mathcal{R}} \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\delta_\omega \varphi_\alpha + \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,j}}\delta_\omega \varphi_{\alpha,j} + \frac{\boldsymbol{\partial}\mathcal{L}}{\boldsymbol{\partial} x^k}\delta_\omega x^k \right\} dx$$

which by (27) becomes

$$= \int_{\mathcal{R}} \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\Delta_\omega \varphi_\alpha + \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,j}}\frac{\partial}{\partial x^j}(\Delta_\omega \varphi_\alpha) \right.$$
$$\left. + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\varphi_{\alpha,k} + \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,j}}\varphi_{\alpha,jk} + \frac{\boldsymbol{\partial}\mathcal{L}}{\boldsymbol{\partial} x^k} \right]}_{= \frac{\partial \mathcal{L}}{\partial x^k}} \delta_\omega x^k \right\} dx$$

so after some slight manipulation we obtain

$$= \int_{\mathcal{R}} \left\{ \delta_\omega x^k \frac{\partial \mathcal{L}}{\partial x^k} + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \frac{\partial}{\partial x^j}\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,j}} \right]}_{0} \Delta_\omega \varphi_\alpha + \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}}\Delta_\omega \varphi_\alpha \right) \right\} dx$$

Here the expression internal to the square bracket *vanishes by virtue of our assumption that φ is dynamical*. Combining this result with that achieved near the bottom of the preceeding page, we obtain

$$\delta_\omega S_{\mathcal{R}}[\varphi] = \int_{\mathcal{R}} \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}}\Delta_\omega \varphi_\alpha + \mathcal{L}\delta_\omega x^k \right] dx$$

Drawing finally upon (26.1) and (27.1), we obtain Noether's Theorem:

$$\delta_\omega S_{\mathcal{R}}[\varphi_{\text{dynamical}}] = \sum_{r=1}^{\nu} \delta\omega^r \cdot \int_{\mathcal{R}} \left(\partial_k J_r^k\right) dx = \sum_{r=1}^{\nu} \delta\omega^r \cdot \int_{\partial\mathcal{R}} J_r^k d\sigma_k \tag{28}$$

with

$$J_r^k = \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}}\left\{ \Phi_{\alpha r} - \varphi_{\alpha,i}\mathcal{X}_r^i \right\} + \mathcal{L}\mathcal{X}_r^k \tag{29}$$

where $\sum_\alpha$ and $\sum_k$ are understood.

Equation (29) can—quite naturally, in view of the construction $\partial_k J_r^k$ which made unbidden claim to our attention at (28)— be considered to describe the k-indexed *components* of an object $\boldsymbol{J}_r$. One such object—one such "Notherian current"—is associated with each of the parameters ω^r which enter into the description of the map $\mathcal{T}$. One can write out the the explicit description

$$J_r^k = J_r^k(\varphi, \partial\varphi, x)$$

of such a $\boldsymbol{J}_r$ as soon as one is in possession of (i) the Lagrangian density $L(\varphi, \partial\varphi, x)$ characteristic of the system in hand, and (ii) the structure functions $\mathcal{X}^i_r(x)$ and $\Phi_{\alpha r}(\varphi, x)$ characteristic of the map. The question, however, remains: what is such knowledge good for?

General considerations relating to the application of Noether's Theorem. We came at (28) to a conclusion of which

$$\delta_\omega S_{\mathcal{R}}[\varphi] = \sum_r \delta\omega^r \cdot \int_{\mathcal{R}} \text{div}\boldsymbol{J}_r dx = \sum_r \delta\omega^r \cdot \int_{\partial\mathcal{R}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} \tag{30}$$

provides a picturesque abbreviation. This is a result of charming simplicity, but it is for the power of its immediate implications that it is celebrated. Suppose, for example, that it could on some grounds be asserted that

$$\delta_\omega S_{\mathcal{R}}[\varphi] = 0 \quad \text{for all bubbles } \mathcal{R} \text{ and all variations } \delta\omega \tag{31}$$

It would then follow that

$$\partial_k J^k_r = 0 \quad : \quad r = 1, 2, \ldots, \nu \tag{32}$$

These are "continuity equations," statements of the form

$$\tfrac{\partial}{\partial t}(\text{density}) + \boldsymbol{\nabla}\cdot(\textbf{flux}) = 0$$

What we have in (32) is an ν-fold set of *conservation laws*.

Insofar as (31) $\Longrightarrow$ (32), Noether's Theorem serves to provide a particularly precise and powerful formulation of the connection between *symmetries* (of the dynamical action) on the one hand, and *conservation laws* on the other. It derives its power in part from the fact that it formulates the association

$$\text{symmetry} \quad \longleftrightarrow \quad \text{conservation}$$

in terms which are rooted in a variational principle, and which are, therefore, essentially coordinate-free. When a new conservation law has been discovered (experimentally, let us say), it becomes urgent in this light to undertake a search for the underlying symmetry, and when such a symmetry is discovered it is difficult to resist the conclusion that one has discovered something "deep."[16]

It is useful to notice that Noether's Theorem gives rise to currents $\boldsymbol{J}$ which tend generally to be "interesting" to precisely the degree that the associated map is interesting—whether or not $\boldsymbol{J}$ happens to be in fact conserved. Energy, momentum, angular momentum... are, interesting (because frequently useful) physical constructs even in contexts where they are not conserved. Special

[16] Such searching may, however, prove futile. Contrary to a widely-held belief, there exist conservation laws which do *not* have their origin in invariance properties of the dynamical action.

interest attaches (but not exclusively) to maps which embody *isometries of the spacetime manifold*.

One should bear in mind that conservation laws—whatever the "symmetry considerations" that may have been that led to their discovery—have ultimately this status: they are *implications of the equations of motion*. This is true even when (as in relativistic field theory) the symmetry is one which has been intentionally "built into" the field equations. By way of illustration, consider the simple "translational map"

$$\mathcal{T}^{\text{translation}}_{\delta\omega} \quad : \quad \begin{cases} x^i \longrightarrow X^i(x\,;\delta\omega) = & x^i + \delta\omega^i \\ \varphi_\alpha(x) \longrightarrow \Phi_\alpha(X;\delta\omega) = & \varphi_\alpha(x) \end{cases} \tag{33}$$

Comparison with (26) shows the associated structure functions to be given by

$$\mathcal{X}^i_j = \delta^i_j \quad \text{and} \quad \Phi_{\alpha j} = 0$$

We are led thus from (29) to expressions of the design

$$J^k_j = \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}} \Big\{ 0 - \varphi_{\alpha,i}\delta^i_j \Big\} + \mathcal{L}\delta^k_j$$

which, in respect for entrenched tradition, we agree to notate

$$S^k{}_j = \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}} \varphi_{\alpha,j} - \mathcal{L}\delta^k{}_j \tag{34}$$

and to call the "stress-energy tensor."[17] Does (33) describe in fact asymmetry, in the sense $\delta_\omega S_{\mathcal{R}}[\varphi] = 0$, of the dynamical action? Is it in fact the case that $\partial_k S^k{}_j = 0$? By calculation

$$\partial_k S^k{}_j = \left[\frac{\partial}{\partial x^k} \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}} \right] \varphi_{\alpha,j} + \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}} \varphi_{\alpha,kj} - \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} \varphi_{\alpha,j} - \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,i}} \varphi_{\alpha,ij} - \frac{\boldsymbol{\partial} \mathcal{L}}{\boldsymbol{\partial} x^k}$$

Since the second term cancels the fourth (trivially), and the first cancels the third *in consequence of the equations of motion*, we have

$$\begin{aligned} \partial_k S^k{}_j &= -\frac{\boldsymbol{\partial} \mathcal{L}}{\boldsymbol{\partial} x^k} \\ &= 0 \quad \text{if and only if } \mathcal{L} \text{ has no } \textit{explicit } x\text{-dependence} \end{aligned}$$

[17] It is, of course, entirely natural to assign particularized names/notations to Noetherian currents which—as here—derive from particularized assumptions. But when one attaches the word "tensor" to an object one is not simply alluding to its indicial decorations; one is making a statement concerning the explicit *transformation properties* of the object in question. We ourselves have yet to discuss the transformation properties of $S^k{}_j$.

In all applications of Noether's Theorem one stands with one foot planted in particularities of the map, and the other in particularities of the system—that is, of the Lagrange density $\mathfrak{L}$ which serves to describe the system. In the preceeding discussion $\mathfrak{L}$ remained unspecified at (34), and we came ultimately to an $\mathfrak{L}$-dependent conclusion. Suppose it were in fact the case that the Lagrangian possessed the x-independent structure

$$\mathfrak{L} = \mathfrak{L}(\varphi, \partial\varphi)$$

which the conservation law(s) $\partial_k S^k{}_j = 0$ have been seen to entail. We are in position now to appreciate the importance of the observation that *by gauge transformation, structural features of the Lagrangian—whence also symmetry properties of the associated action functional—can be profoundly altered*. The Lagrangian

$$\mathfrak{L}' = \mathfrak{L}(\varphi, \partial\varphi) + \partial_k \mathcal{G}^k(\varphi, x) = \mathfrak{L}'(\varphi, \partial\varphi, x)$$

will, in general, *not* be x-independent; it serves equally well to describe the physical system in hand (it gives rise to the same field equations), but leads *via* (34) to an $S'^k{}_j$ which is distinct from $S^k{}_j$ and which is, in general, *not* conserved. Had we adopted $\mathfrak{L}'$ at the outset, we would have obtained $\partial_k S'^k{}_j \neq 0$, and would—though they remain valid properties of the system—have *missed* the conservation laws $\partial_k S^k{}_j = 0$. We would have picked up the latter information only if we had thought to ask

Can the x-dependence of $\mathfrak{L}'(\varphi, \partial\varphi, x)$ be "gauged away"?

As was observed already at (23)

$$\mathfrak{L} \longrightarrow \mathfrak{L}' = \mathfrak{L} + \partial_k \mathcal{G}^k \quad \text{induces} \quad S_{\mathcal{R}} \longrightarrow S'_{\mathcal{R}} = S_{\mathcal{R}} + \int_{\partial\mathcal{R}} \mathcal{G}^k d\sigma_k \tag{35}$$

It is the boundary term which, though invisible to Hamilton's Principle, can do violence to applications of Noether's Theorem. Reading from (29), we obtain

$$J^k_r \longrightarrow J'^k_r = J^k_r + \underbrace{\left\{ \left[\Phi_{\alpha r} - \varphi_{\alpha,i} \mathcal{X}^i_r\right] \frac{\partial}{\partial \varphi_{\alpha,k}} + \mathcal{X}^k_r \right\} \left(\partial_j \mathcal{G}^j\right)}_{G^k_r(\varphi, \partial\varphi, x)} \tag{36}$$

In exceptional cases it will be possible to write

$$G^k_r = \partial_j A^{jk}_r \quad \text{with} \quad A^{jk}_r = -A^{kj}_r$$

In such cases—some of which are, as will emerge, physically quite important—the symmetry of the action functional is preserved; one has

$$\partial_k J^k_r = 0 \quad \Longleftrightarrow \quad \partial_k J'^k_r = 0$$

It becomes natural in this light to anticipate that there will arise cases in which it is appropriate to absorb "parameterized gauge transformations" into an enlarged conception of what we are to mean by a "parameterized map," writing

$$\mathcal{T}_{\delta\omega} \quad : \quad \left\{ \begin{array}{rcl} x^i & \longrightarrow & x^i + \delta\omega^r \cdot \mathcal{X}_r^i \\ \varphi_\alpha(x) & \longrightarrow & \varphi_\alpha(x) + \delta\omega^r \cdot \Phi_{\alpha r} \\ \mathcal{L} & \longrightarrow & \mathcal{L} + \delta\omega^r \cdot \partial_j \mathcal{G}_r^j \end{array} \right. \tag{37}$$

in place of (25). Slight adjustment of the argument that gave (29) then gives

$$J_r^k = \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,k}} \Big\{ \Phi_{\alpha r} - \varphi_{\alpha,i} \mathcal{X}_r^i \Big\} + \mathcal{L}\mathcal{X}_r^k + \mathcal{G}_r^k \tag{38}$$

We recall in this connection that in particle mechanics it is precisely such a generalization that makes it possible to construct a Noetherian account of the implications of Galilean covariance.[18]

Each of the statements (32) provides what is, in effect, the differential formulation of a *local conservation law*. What are the associated "conserved quantities?" The question is best approached by looking to the corresponding integral statements

$$\int_{\partial\mathcal{R}} J_r^k d\sigma_k = 0 \quad \text{for all } \mathcal{R}, \text{ with } r = 1, 2, \ldots, \nu \tag{39}$$

Take $\mathcal{R}$ to have, in particular, the form of a "spacetime drum," as illustrated in the figure at the top of the next page. We then have

$$\int_{\text{top}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} + \int_{\text{sides}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} + \int_{\text{bottom}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} = 0$$

The middle term will be assumed to vanish, either because we have imposed spatial boundary conditions of the form $\boldsymbol{J}_r(\text{sides}) = \boldsymbol{0}$ or because we have "pushed the sides of the drum to infinity," where $\boldsymbol{J}_r$ has been assumed to die a natural asymptotic death. The surface differentials $\boldsymbol{d\sigma}$ are, by stipulation of the divergence theorem, all "outer-directed," which on the bottom of the drum means "past-directed." Let us, however, adopt the convention that *surface differentials associated with "timeslices" (surfaces of constant t) will in all cases be "future-directed."* We then have

$$\int_{\text{top}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} + 0 - \int_{\text{bottom}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} = 0$$

or again

$$\int_{\text{top}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} = \int_{\text{bottom}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma}$$

—quite irrespectively of the particular t-value used to position the top of the

[18] See, for example, CLASSICAL MECHANICS (1983), p. 169 and the discussion which appears on pp. 161–170 of CLASSICAL FIELD THEORY (1979).

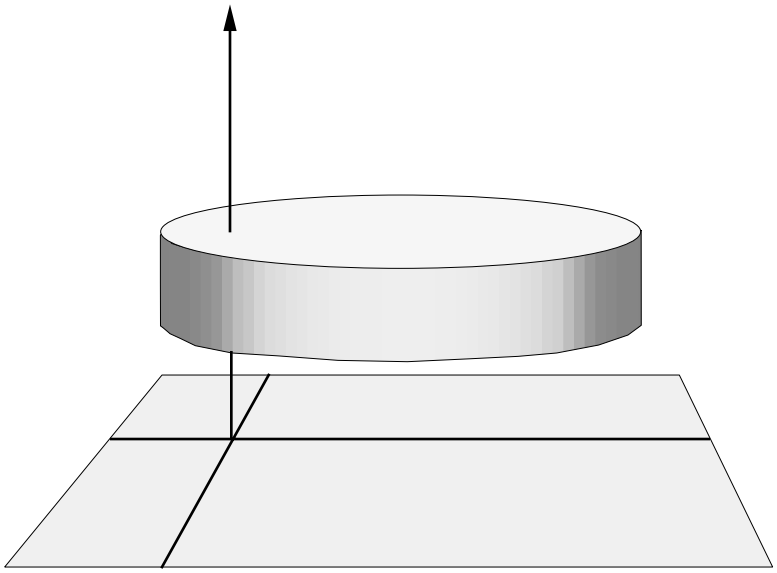


FIGURE 6: *A "drum" in spacetime—a bubble bounded above and below by "timeslices." All points on the top surface have time coordinate t, and all points on the bottom have time coordinate t_0. The edges of the drum may, at the end of the argument, recede to infinity.*

drum. The implication is that the integrated expressions

$$\int_{\text{top}} \boldsymbol{J}_r \cdot \boldsymbol{d\sigma} \equiv \iint \cdots \int J_r^0 dx^1 dx^2 \cdots dx^n \quad : \quad r = 1, 2, \ldots, \nu \tag{40}$$

are (global) *constants of the field motion*.

Returning, by way of illustration, to the translational map (33), we learn from (34) that

$$\left.\begin{aligned} S^0{}_0 &= \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,0}} \varphi_{\alpha,0} - \mathcal{L} \equiv \mathcal{E} \\ S^0{}_1 &= \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,0}} \varphi_{\alpha,1} \qquad \equiv \mathcal{P}_1 \\ S^0{}_2 &= \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,0}} \varphi_{\alpha,2} \qquad \equiv \mathcal{P}_2 \\ S^0{}_3 &= \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha,0}} \varphi_{\alpha,3} \qquad \equiv \mathcal{P}_3 \end{aligned}\right\} \tag{41}$$

Here I have, in the interests of physical concreteness, set $n = 3$; I have assigned x^0 the meaning $x^0 \equiv t$ and have understood $\{x^1, x^2, x^3\}$ to refer to an inertial Cartesian frame in physical 3-space. We note that $\mathcal{E}$ is co-dimensional with $\mathcal{L}$, and has the dimensionality therefore of an *energy density* (energy/volume), while $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ each has the dimensionality (energy density/velocity)

of *linear momentum density*. We are in position now to assert that if the Lagrangian density has no explicit x-dependence (i.e., is invariant with respect to translations in spacetime), then the following number-valued expressions are global constants of the field motion:

$$E = \int\int\int \mathcal{E}\ dx^1dx^2dx^3 = \text{total energy}$$
$$P_1 = \int\int\int \mathcal{P}_1\, dx^1dx^2dx^3 = \text{total 1-component of linear momentum}$$
$$P_2 = \int\int\int \mathcal{P}_2\, dx^1dx^2dx^3 = \text{total 2-component of linear momentum}$$
$$P_3 = \int\int\int \mathcal{P}_3\, dx^1dx^2dx^3 = \text{total 3-component of linear momentum}$$

The local equations $\partial_k J^k_r = 0$ can now be rendered

$$\tfrac{\partial}{\partial t}(\text{energy density } \mathcal{E}) + \boldsymbol{\nabla}\cdot(\text{energy } \mathbf{flux}) = 0$$
$$\tfrac{\partial}{\partial t}(\text{momentum density } \mathcal{P}_1) + \boldsymbol{\nabla}\cdot(\text{associated momentum } \mathbf{flux}) = 0$$
$$\tfrac{\partial}{\partial t}(\text{momentum density } \mathcal{P}_2) + \boldsymbol{\nabla}\cdot(\text{associated momentum } \mathbf{flux}) = 0$$
$$\tfrac{\partial}{\partial t}(\text{momentum density } \mathcal{P}_3) + \boldsymbol{\nabla}\cdot(\text{associated momentum } \mathbf{flux}) = 0$$

and by straightforward adjustment of the arguments that gave (41) we can obtain *explicit descriptions* of the fluxes in question. These statements assign explicitly detailed meaning to the statement that field energy and momentum, when globally conserved, are conserved because they slosh about in a locally conservative way. And—to restate a point already made—the total energy and momentum of a field system are of manifest "interest" even when they happen *not* to be conserved!

One final remark: at no point in our work thus far have we made actual use of the "group structure" which has been presumed to attach to the infinitesimal parameterized maps which are themselves clearly central to Noether's line of argument. The families of transformations which lay natural claim to our attention do tend generally—spontaneously—to possess the group property, but nowhere have we had to draw upon any of the rich consequences of that fact. That situation will change when we look to details pursuant to certain (important) specific *applications* of Noether's Theorem.

Field-theoretic analog of the Helmholtz conditions. Generalized forces $F_i(q)$ which are "conservative" in the sense they can be derived from a potential

$$F_i(q) = -\frac{\partial}{\partial q^i}U(q)$$

have (owing to the general *equality of the cross derivatives* of $U(q)$) necessarily the property that

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} = 0$$

Conversely (by a famously more difficult line of argument), if $F_i(q)$ possesses the latter property then there exists such a function $U(q)$; it is in fact the case that—in particular consequence of a very general formula due to Poincaré[19]—$U(q)$ admits of this little-known but wonderful explicit construction

$$U(q) = -\int_0^1 F_k(\tau q) q^k d\tau + \text{constant} \tag{42}$$

The preceeding remarks serve to generalize (very slightly) the familiar statement

$$\boldsymbol{A} = \text{grad}\varphi \quad \Longleftrightarrow \quad \text{curl}\boldsymbol{A} = \boldsymbol{0}$$

Identical ideas enter into the observation[20] that the dynamical system

$$\begin{aligned} \dot{q} &= f(q,p) \\ \dot{p} &= g(q,p) \end{aligned}$$

will admit of Hamiltonian formulation if and only if it is true of the functions $f(q,p)$ and $g(q,p)$ that

$$\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} = 0$$

It seems entirely natural, in the light of such remarks, to ask a question which—quite unaccountably to me—appears in fact to be only very seldom asked: Under what conditions do the coupled second-order differential equations

$$\begin{aligned} G_1(\ddot{q}, \dot{q}, q, t) &= 0 \\ G_2(\ddot{q}, \dot{q}, q, t) &= 0 \\ &\vdots \\ G_n(\ddot{q}, \dot{q}, q, t) &= 0 \end{aligned}$$

admit of Lagrangian formulation

$$G_i(\ddot{q}, \dot{q}, q, t) = \left\{ \frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i} \right\} L(\dot{q}, q, t)$$

The question was first explored by Hermann von Helmholtz (1821– 1894), who in 1867 established the necessity of the conditions

$$\left.\begin{aligned} \frac{\partial G_i}{\partial \ddot{q}^j} - \frac{\partial G_j}{\partial \ddot{q}^i} &= 0 \\ \frac{\partial G_i}{\partial \dot{q}^j} + \frac{\partial G_j}{\partial \dot{q}^i} &= \frac{d}{dt}\left[\frac{\partial G_i}{\partial \ddot{q}^j} + \frac{\partial G_j}{\partial \ddot{q}^i}\right] \\ \frac{\partial G_i}{\partial q^j} - \frac{\partial G_j}{\partial q^i} &= \frac{1}{2}\frac{d}{dt}\left[\frac{\partial G_i}{\partial \dot{q}^j} - \frac{\partial G_j}{\partial \dot{q}^i}\right] \end{aligned}\right\} \tag{43}$$

[19] See Electrodynamics (1972), p. 173, or p. 14 of my "Electrodynamical Applications of the Exterior Calculus" (1996).

[20] See Classical Mechanics (1983), p. 209.

for which A. Mayer in 1896 established the sufficiency. To establish necessity one has simply to notice that differential equations derived from a Lagrangian are differential equations

$$\begin{aligned} G_i(\ddot{q},\dot{q},q,t) &= \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\ddot{q}^j + \frac{\partial^2 L}{\partial\dot{q}^i\partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial\dot{q}^i\partial t} - \frac{\partial L}{\partial q^i} \\ &= g_{ij}(\dot{q},q,t)\ddot{q}^j + h_i(\dot{q},q,t) \end{aligned}$$

of a very particular structure; they are, for example, linear in the 2nd derivatives $\ddot{q}^j$, and the coefficients g_{ij} are necessarily symmetric. The Helmholtz conditions (43) emerge quite naturally when such observations are collected and—this is the point—formulated in such a way as to make no explicit reference to the (generally unknown) Lagrangian itself.[21] In (43) we are presented with an antisymmetric array + a symmetric array + another antisymmetric array of conditions—conditions which in number total

$$\tfrac{1}{2}(n-1)n + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n-1)n = \tfrac{1}{2}n(3n-1) = 1,5,12,22,35,\ldots \sim \tfrac{3}{2}n^2$$

The practical utility of the Helmholtz conditions is limited however not so much by their number as by the fact that we never know whether equations that fail the test might by appropriate "pre-processing" be made to pass it; by the converse, that is to say, of the following observation: if equations $G_i = 0$ pass the test, then reorderings, multiplication by factors, formation of linear combinations, etc. will result generally in *equivalent* equations that nevertheless *fail* the test. This awkward circumstance has motivated P. Havas[22] to pose and resolve this more general question: Given an ordered system of equations

$$G_i(\ddot{q},\dot{q},q,t) = 0$$

when do there exist integrating factors $f_i(\dot{q},q,t)$ such that the equivalent system

$$\tilde{G}_i(\ddot{q},\dot{q},q,t) \equiv f_i(\dot{q},q,t)\cdot G_i(\ddot{q},\dot{q},q,t) = 0$$

admit of Lagrangian formulation? Unsurprisingly, the conditions achieved by Havas are markedly more complicated than the Helmholtz conditions. And the Havas conditions, for all their complexity, contribute nothing toward the resolution either of the ordering problem or of the linear combination problem.

Look, by way of illustration, to the simple system

$$G(\ddot{q},\dot{q},q,t) \equiv \ddot{q} + q = 0$$

Here $n = 1$; there is a single Helmholtz conditon, it reads

$$\frac{\partial G}{\partial\dot{q}} = \frac{d}{dt}\left[\frac{\partial G}{\partial\ddot{q}}\right]$$

[21] For the details see pp. 117–120 of Classical Mechanics (1983).

[22] "The range of application of the Lagrange formalism-I," Nuovo Cimento Supp. **5**, 363 (1957).

and is clearly satisfied; there exists an associated Lagrangian (Helmholtz does not tell us how to find it) and by familiar tinkering we know it to be

$$L(\dot{q}, q, t) = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}q^2 + \text{possible gauge term}$$

Look next to the system

$$G(\ddot{q}, \dot{q}, q, t) \equiv \ddot{q} + k\dot{q} + q = 0$$

The single Helmholtz condition now entails $k = 0$, which is the case already studied. We conclude that if $k \neq 0$ then no Lagrangian exists. Suppose, however, we ask this weaker question: Does there exist an integrating factor $f(t)$ such that the equivalent equation

$$\tilde{G}(\ddot{q}, \dot{q}, q, t) \equiv f(t) \cdot G(\ddot{q}, \dot{q}, q, t) = f \cdot (\ddot{q} + k\dot{q} + q) = 0$$

admits of Lagrangian formulation? The Helmholtz condition is seen now to entail $fk = \dot{f}$. We conclude that if $f(t) = f_0 e^{kt}$ then $\tilde{G}(\ddot{q}, \dot{q}, q, t) = 0$ *does* admit of Lagrangian formulation (therefore of Hamiltonian formulation, therefore even of quantum mechanical formulation!), and by unfamiliar tinkering discover the Lagrangian to be given by

$$L(\dot{q}, q, t) = \tfrac{1}{2} f_0 e^{kt}(\dot{q}^2 - q^2) + \text{possible gauge term}$$

We recover the previous Lagrangian at $k = 0$, provided we set the physically inconsequential prefactor f_0 equal to unity. By exercise of some uncommon self-control, I shall forego discussion of some of the interesting physics that can be extracted from generalizations of this striking result.

One gains the impression that from some sufficiently exhaulted formal standpoint the Helmholtz conditions can be understood as but yet another instance of the familiar "curl condition." The reader who wishes to look more closely into that or other aspects of our present topic might be well-advised to start by looking into the dense pages of R. Santilli's *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics* (1978).

Returning now to field theory, we find it natural, in light of the preceeding discussion, to ask: Under what conditions do the coupled second-order partial differential equations

$$\begin{aligned} G_1(\partial\partial\varphi, \partial\varphi, \varphi, x) &= 0 \\ G_2(\partial\partial\varphi, \partial\varphi, \varphi, x) &= 0 \\ &\vdots \\ G_N(\partial\partial\varphi, \partial\varphi, \varphi, x) &= 0 \end{aligned}$$

admit of Lagrangian formulation

$$G_\alpha(\partial\partial\varphi, \partial\varphi, \varphi, x) = \left\{ \frac{\partial}{\partial x^i} \frac{\partial}{\partial \varphi_{\alpha,i}} - \frac{\partial}{\partial \varphi_\alpha} \right\} \mathcal{L}(\partial\varphi, \varphi, x)$$

Havas, in the introduction to his paper, remarks that "the general problem was studied in great detail by Königsberger,[23] *who also investigated continuous systems*" [my emphasis], so it seems quite possible that the result I am about to describe was known to L. Königsberger and his readers (who, curiously, seem not to have included E. T. Whittaker among their number) already 78 years before it was worked out by me. In any event... if one proceeds in direct imitation of the line of argument which led to (43) one is led[24] to conditions which can be notated

$$\left.\begin{aligned} \frac{\eth G_\alpha}{\eth \varphi_{\beta,ij}} - \frac{\eth G_\beta}{\eth \varphi_{\alpha,ij}} &= 0 \\ \frac{\partial G_\alpha}{\partial \varphi_{\beta,j}} + \frac{\partial G_\beta}{\partial \varphi_{\alpha,j}} &= \frac{\partial}{\partial x^i}\left[\frac{\eth G_\alpha}{\eth \varphi_{\beta,ij}} + \frac{\eth G_\beta}{\eth \varphi_{\alpha,ij}}\right] \\ \frac{\partial G_\alpha}{\partial \varphi_\beta} - \frac{\partial G_\beta}{\partial \varphi_\alpha} &= \frac{1}{2}\frac{\partial}{\partial x^i}\left[\frac{\partial G_\alpha}{\partial \varphi_{\beta,i}} - \frac{\partial G_\beta}{\partial \varphi_{\alpha,i}}\right] \end{aligned}\right\} \tag{44}$$

and concerning which my first obligation is to clarify the meaning and origin of the fancy derivatives. It is an implication of $\varphi_{\alpha,ij} = \varphi_{\alpha,ji}$ that

$$\varphi_{\alpha,ij} = (1-\lambda)\varphi_{\alpha,ij} + \lambda\varphi_{\alpha,ji} \quad (\text{all } \lambda)$$

and follows therefore that

$$F(\ldots, \varphi_{\alpha,ij}, \ldots, \varphi_{\alpha,ji}, \ldots)$$

and its substitutional transform

$$\begin{aligned} &\tilde{F}(\ldots, \varphi_{\alpha,ij}, \ldots, \varphi_{\alpha,ji}, \ldots) \\ &\quad \equiv F(\ldots, (1-\lambda)\varphi_{\alpha,ij} + \lambda\varphi_{\alpha,ji}, \ldots, (1-\mu)\varphi_{\alpha,ji} + \mu\varphi_{\alpha,ij}, \ldots) \end{aligned}$$

are in all cases equal, but (in most cases) functionally distinct. The definition

$$\frac{\eth}{\eth \varphi_{\alpha,ij}} \equiv \frac{1}{2}\left[\frac{\partial}{\partial \varphi_{\alpha,ij}} + \frac{\partial}{\partial \varphi_{\alpha,ji}}\right]$$

has been cooked up to achieve (in all cases, even—trivially—in the case $i = j$)

$$\frac{\eth}{\eth \varphi_{\alpha,ij}} F = \frac{\eth}{\eth \varphi_{\alpha,ij}} \tilde{F}$$

and thus to yield results which are invariant with respect to the exercise of our substitutional options. If N signifies the number of field components, and

[23] *Die Principien der Mechanik* (1901).

[24] See Classical Field Theory (1979), p. 121–124.

$m = n + 1$ the number of independent variables, then by delicate counting we find the conditons (44) to be

$$\begin{aligned}\tfrac{1}{2}(N-1)N \cdot \tfrac{1}{2}m(m+1) + \tfrac{1}{2}N(N+1)\cdot m + \tfrac{1}{2}(N-1)N \\ = \tfrac{1}{4}N\big[N(m^2+3m+2) - (m^2-m+2)\big]\end{aligned}$$

in number. We therefore have

$$\begin{array}{lll}\tfrac{1}{2}N(\ 3N-\ 1) & \text{conditions if} & m=1\\ \tfrac{1}{2}N(\ 6N-\ 2) & \text{conditions if} & m=2\\ \tfrac{1}{2}N(10N-\ 4) & \text{conditions if} & m=3\\ \tfrac{1}{2}N(15N-\ 7) & \text{conditions if} & m=4\\ \tfrac{1}{2}N(21N-11) & \text{conditions if} & m=5\end{array}$$

and recover precisely the Helmholtz conditions (43) in the case $m = 1$. The conditions (44) are beset with all the limitations which have previously been seen to afflict the practical application of the Helmholtz conditions (and are susceptible, I suppose, to the same modes of potential remedy). They are necessary by demonstration, but concerning their sufficiency one can, at this point, only speculate.

Look, by way of illustration, to the class of simple systems

$$G(\partial\partial\varphi, \partial\varphi, \varphi, x) = \varphi_{tt} - \varphi_{xx} + k\varphi^p = 0$$

Here $N = 1$ and $m = 2$; there are two conditions (44), and they read

$$\begin{aligned}\frac{\partial G}{\partial \varphi_t} &= \frac{\partial}{\partial t}\left[\frac{\eth G}{\eth \varphi_{tt}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G}{\eth \varphi_{xt}}\right]\\ \frac{\partial G}{\partial \varphi_x} &= \frac{\partial}{\partial t}\left[\frac{\eth G}{\eth \varphi_{tx}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G}{\eth \varphi_{xx}}\right]\end{aligned}$$

Both are readily seen to be satisfied, so we are entitled to hope (yet cannot, in the absence of a sufficienty proof, be certain) that the system admits of Lagrangian formulation. A little tinkering yields

$$\mathcal{L}(\partial\varphi, \varphi) = \tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2 - \tfrac{1}{p+1}k\varphi^{p+1}$$

At $k = 0$ we recover the essentials (compare (15)) of the familar wave equation, and at $p = 1$ we obtain the Lagrangian system

$$\varphi_{tt} - \varphi_{xx} + k\varphi = 0$$

which will acquire importance for us as the "Klein-Gordon equation." By easy extension we have

$$\mathcal{L}(\partial\varphi, \varphi) = \tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2 - F(\varphi) \quad \Longleftrightarrow \quad \varphi_{tt} - \varphi_{xx} + F'(\varphi) = 0$$

In the particular case $F(\varphi) = -k\cos\varphi$ we obtain a much-studied nonlinear field equation

$$\varphi_{tt} - \varphi_{xx} + k\sin\varphi = 0$$

known as the "Sine-Gordon equation," which in the weak-field approximation gives back the (linear) Klein-Gordon equation.

Look next to the system

$$G(\partial\partial\varphi, \partial\varphi, \varphi, x) = \varphi_t - \varphi_{xx} = 0$$

which captures the essence of the so-called "heat equation" (otherwise known as the "diffusion equation"). From the same pair of conditions as served us in the preceeding example we are led promptly to the conclusion that the heat equation does *not* admit of Lagrangian formulation. Nor does the obvious variant of the "integrating factor trick" salvage the situation. That I would encourage my reader to regard as a profoundly unsatisfactory state of affairs, as an invitation to invention. For the heat equation is an important thing, and the Lagrangian formalism is an important thing, and it is "unreasonable" that they should have nothing to say to each other. Since we will fairly frequently find ourselves in analogous predicaments, I digress to illustrate the kind of escape routes that can, with a little cleverness, be devised. One standard trick—the "auxiliary field trick," as it is sometimes called—hinges on the recognition that our field may possess a heretofore overlooked "companion field." With such an idea in mind, it does in the present context not take one long to concoct the 2-field system

$$\mathcal{L} = \tfrac{1}{2}(\alpha_t\varphi - \alpha\varphi_t) - \alpha_x\varphi_x$$

and to observe that

$$\left\{\frac{\partial}{\partial t}\frac{\partial}{\partial \alpha_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \alpha_x} - \frac{\partial}{\partial \alpha}\right\}\mathcal{L} = 0 \quad \text{gives} \quad \varphi_{xx} = +\varphi_t$$

$$\left\{\frac{\partial}{\partial t}\frac{\partial}{\partial \varphi_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \varphi_x} - \frac{\partial}{\partial \varphi}\right\}\mathcal{L} = 0 \quad \text{gives} \quad \alpha_{xx} = -\alpha_t$$

We have successfully reproduced the diffusion equation, but at the cost of introducing an auxiliary field that satisfies an "antidiffusion equation," and the question becomes "What are we to make of that?" Have we made a discovery, or merely replaced one embarrassment by another? Such questions tend generally to be important questions, but to admit of no easy (or at least of no universal) answer; everything hinges on details of the particular system at hand.[25] The natural force of well-crafted formalism is frequently more persuasive—and a surer guide—than the imperfect evidence of a laboratory. But tact and good

[25] Later I will have occasion to develop the sense in which manipulations quite similar to those just sketched lead with a kind of inevitability to "the invention of quantum mechanics!" One would appear to be well within one's rights to say of such a development that it is "important," and has the character of a "discovery."

judgment are all-important, for the free-spinning formalist teeters always on the edge of triviality. For example:

Allegations of the form $A = B$ can—whether true or false—always be displayed as "variational principles"

$$(A - B)^2 = \text{minimum}$$

but it would be frivilous to claim that the elegance of the display enhances the likelihood that the allegation is correct.[26] Somewhat less frivilous is the observation that, while (as we have seen) the system

$$\begin{aligned}\dot{q} &= f(q,p)\\ \dot{p} &= g(q,p)\end{aligned}$$

admits of Hamiltonian formulation only when a certain "curl condition" is satisfied, it can always be absorbed into an expanded system which does admit of such formulation: construe q and p to be "coordinates in a 2-space," let Q and P denote their conjugate momenta, and construct

$$H(q,Q,p,P) = Qf(q,p) + Pg(q,p)$$

Then

$$\begin{aligned}\dot{q} &= +\partial H/\partial Q = f(q,p)\\ \dot{p} &= +\partial H/\partial P = g(q,p)\end{aligned}$$

which have, however, acquired these companions:

$$\begin{aligned}\dot{Q} &= -\partial H/\partial q = -Q\frac{\partial f}{\partial q} - P\frac{\partial g}{\partial q}\\ \dot{P} &= -\partial H/\partial p = -Q\frac{\partial f}{\partial p} - P\frac{\partial g}{\partial p}\end{aligned}$$

It's an admittedly "cheap trick," but on occasion proves useful. Similarly "cheap" is the observation that if we wish to achieve

$$G(\partial\partial\varphi, \partial\varphi, \varphi, x) = 0$$

and are not opposed to the free introduction of auxiliary fields, then to achieve contact with the Lagrangian formalism we have only to constuct

$$\mathcal{L}(\partial\partial\varphi, \partial\varphi, \varphi, \alpha, x) = -\alpha G(\partial\partial\varphi, \partial\varphi, \varphi, x)$$

Trivially,

$$\left\{\frac{\partial}{\partial x^i}\frac{\partial}{\partial\left(\frac{\partial\alpha}{\partial x^i}\right)} - \frac{\partial}{\partial\alpha}\right\}\mathcal{L} = G(\partial\partial\varphi, \partial\varphi, \varphi, x) = 0$$

[26] See, however, the discussion of Gauß' "Principle of Least Constraint" which can be found in §105 of Whittaker's *Analytical Mechanics*.

but we acquire an obligation to describe also the motion of the auxiliary field, and that (since the Lagrangian depends now—non-standardly—also upon the *second* partials of φ) requires that we subject the Lagrangian formalism to a bit of stretching, along lines first explored in (1850) by M. Ostrogradsky.[27] Thus are we led at length to write (if I may lapse for a moment into the notation appropriate to the descripton of a *multi-component* field system $\varphi = \{\varphi_\alpha\}$)

$$\left\{ -\frac{\partial^2}{\partial x^i \partial x^j}\frac{\partial}{\partial \varphi_{\alpha,ij}} + \frac{\partial}{\partial x^i}\frac{\partial}{\partial \varphi_{\alpha,i}} - \frac{\partial}{\partial \varphi_\alpha} \right\} \mathfrak{L} = 0 \tag{45}$$

where terms containing $\varphi_{\alpha,ijk}$ will be avoided if and only if $\varphi_{\alpha,ij}$ enters at most linearly into the structure of the functions $G(\partial\partial\varphi, \partial\varphi, \varphi, x)$. Returning again to our most recent example, we construct

$$\mathfrak{L}' = -\alpha(\varphi_t - \varphi_{xx})$$

and, in addition to the diffusion equation, recover (by application of (45)) precisely the backwards diffusion equation $\alpha_t + \alpha_{xx} = 0$. Nor is this, in fact, surprising, for

$$\begin{aligned} \mathfrak{L} - \mathfrak{L}' &= \tfrac{1}{2}\alpha_t\varphi + \tfrac{1}{2}\alpha\varphi_t - \alpha_x\varphi_x - \alpha\varphi_{xx} \\ &= (\tfrac{1}{2}\alpha_t\varphi)_t + (-\alpha\varphi_x)_x \end{aligned}$$

shows that $\mathfrak{L}$ and $\mathfrak{L}'$ are in fact *gauge equivalent*. Here a "seeming triviality" and a "stroke of modest genius" are seen to be actually of identical force. One should perhaps not be too casually dismissive of trivialities.

Hamiltonian methods in classical field theory. We wrote out the Lagrangian theory of a one-dimensional crystal, traced that theory to its continuous limit where we obtained a Lagrangian description of the dynamics of a string, and by straightforward generalization we led to an elegantly functional "Lagrangian formulation of the classical dynamics of field systems." Plausibly that same strategy would lead us to a "Hamiltonian fomulation of classical field theory," to the associated "theory of canonical transformations," whence finally to a "field-theoretic generalization of Hamilton-Jacobi theory." Alternatively, we might attempt to work entirely within a field-theoretic framework—building upon the Lagrangian formalism in direct imitation of particle-theoretic practice, but avoiding all reference to "crystals," to "refinement of the lattice." We might expect to be led ultimately—by natural extension of standard quantization procedures—to a "quantum theory of fields."

Central to all such formal elaboration, we can anticipate, will be a field-theoretic analog of the "conjugate momentum" concept. Were we to proceed by the lattice-refinement technique, we can expect to achieve[28]

$$p_i = p_i(\dot{q}, q, t) \equiv \frac{\partial L}{\partial \dot{q}^i} \xrightarrow[\text{lattice refinement}]{} \pi_\alpha = \pi_\alpha(\partial\varphi, \varphi, x) \equiv \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}^\alpha}$$

[27] See Whittaker's §110.

[28] At this point it becomes formally more natural to write φ^α where formerly we have written φ_α.

And there's the rub. Field theory presents us with a *population* of co-equal constructions

$$\frac{\partial \mathfrak{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial t}\right)},\ \frac{\partial \mathfrak{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^1}\right)},\ \frac{\partial \mathfrak{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^2}\right)},\ \cdots,\ \frac{\partial \mathfrak{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^n}\right)}$$

and it would appear to be in violation of the essential spirit of field theory (and certainly in violation of the spirit of relativity) to promote one of those—equivalently, to promote one of the independent variables $\{t, x^1, x^2, \ldots, x^n\}$—to a status of preeminence over the others. What to do? Should we associate with each field component an indexed *set* of "conjugate momenta"

$$\varphi^\alpha \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \pi_{\alpha 0} = \frac{\partial \mathfrak{L}}{\partial(\partial \varphi^\alpha / \partial x^0)} \\ \pi_{\alpha 1} = \frac{\partial \mathfrak{L}}{\partial(\partial \varphi^\alpha / \partial x^1)} \\ \qquad \vdots \\ \pi_{\alpha n} = \frac{\partial \mathfrak{L}}{\partial(\partial \varphi^\alpha / \partial x^n)} \end{array} \right.$$

Such a procedure would appear to do such radical violence to the essentials of Hamiltonian mechanics as to be unworkable. Should the $\pi_{\alpha i}$ be made to participate co-equally in the assembly into some unitary object along (perhaps) these general lines

$$\pi_\alpha = \sum_{i=0}^{n} \pi_{\alpha i} P^i$$

To do so would be to import into the theory an auxiliary object P^i which is unprecidented in the Hamiltonian mechanics of particles, and of which no natural candidate presents itself.

By way of preparaion for what follows, I digress now to observe that one can bring geometrical imagery to $\phi(x, y, z)$ (which I shall, in service of concreteness, assume to be number-valued and real) in a variety of distinct ways; one can consider that $\phi(x, y, z)$ describes

a "point" in an (∞-dimensional) space of functions $f(x, y, z)$

a x-parameterized "curve" in a space of functions $f(y, z)$
a y-parameterized "curve" in a space of functions $f(x, z)$
a z-parameterized "curve" in a space of functions $f(x, y)$

a (x, y)-parameterized "curve" in a space of functions $f(z)$
a (x, z)-parameterized "curve" in a space of functions $f(y)$
a (y, z)-parameterized "curve" in a space of functions $f(x)$

a (x, y, z)-parameterized point on the real line

Lagrangian field theory proceeds implicitly from the first point of view, though when we, as physicists, undertake to comprehend what $\varphi(t, x, y)$ is telling us (or to discover the implications of prescribed initial data) we frequently have recourse to the second viewpoint: we make mental "movies." The Hamiltonian formalism(s) developed below embrace the second viewpoint explicitly, from the outset.

Let $\{x^0, x^1, \ldots, x^n\}$ refer to some specified coordinatization of spacetime (of which there are, of course, infinitely many). Identically structured and physically equivalent Hamiltonian formalisms $\mathcal{H}^i$ of $(n+1)$ distinct "flavors" come into existence as follows: In $\mathcal{H}^i$ the variable x^i has been promoted to a distinguished status; it rules as "the parameter." It is the business of $\mathcal{H}^i$ to inscribe "x^i-parameterized dynamical flow curves" in the functional analog Γ^i of a $2N$-dimensional phase space, as illustrated in the following figure. Each

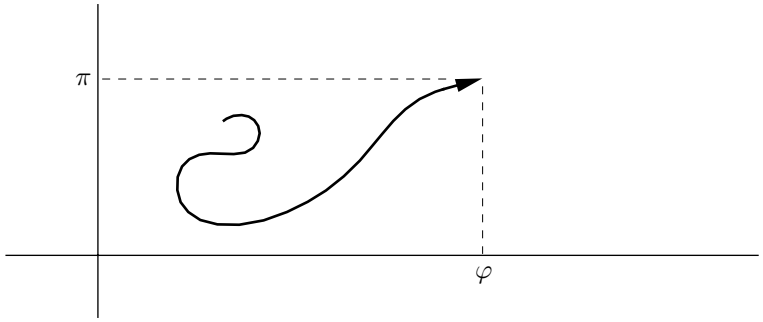


FIGURE 7: *The figure provides a highly schematic representation of a* "dynamical flow curve in the phase space Γ^i." *In the figure,* φ *refers to the* N*-tuple of field functions* $\varphi^\alpha(x)$*, construed to be functions of the variables*

$$x^0, x^1, x^2, \ldots, \bullet, \ldots, x^n \quad \textit{(the missing variable is } x^i\textit{)}$$

and of the parameter x^i*, while* π *refers to the conjugate fields*

$$\pi_\alpha(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial\varphi^\alpha/\partial x^i)}$$

—similarly construed.

individual $\mathcal{H}^i$ proceeds from a symmetry-breaking act—the promotion of an arbitrarily selected variable to "distinguished status." But construction of the composit formalism

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^n$$

entails *no* such symmetry-breaking act. It becomes attractive in this light to regard the field-theoretic "Hamiltonian formalism" to reside in $\mathcal{H}$; to do so would be to embrace a formalism with many redundant component parts, and to accept an obligation ultimately to discuss the (transformation-theoretic and other) interconnections amongst those parts.

In the discussion that follows I shall, in place of x^i, write t to denote the promoted variable. The parameter t can—and in the literature[29] typically

[29] See, for example, H. Goldstein, *Classical Mechanics* (2nd edition 1980), §12–4.

does—refer physically to "time," but should in the generic case be taken to refer to some word like "typical." The generic instance of Hamiltonian field theory will be denoted $\mathcal{H}^t$. Field derivatives with respect to the parameter t will be written $\dot{\varphi}$, while derivatives with respect to the independent variables $\{x^1, x^2, \ldots, x^n\}$ will collectively be denoted ∇x. Where formerly we were content to write $\mathcal{L}(\partial\varphi, \varphi, x)$ we would, by these more explicit conventions, write $\mathcal{L}(\dot{\varphi}, \nabla\varphi; \varphi; t, x)$. We will soon come to attach interest to the observation that the field equations themselves, by these conventions, admit of the following curious reformulation:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\alpha} - \left\{ \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \frac{\partial}{\partial x^k} \sum_{k-1}^{n} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^k}\right)} \right\} = 0$$

Equivalently (since $\mathcal{L}$ has been assumed not to depend upon second derivatives of the fields, and is in particular therefore $\nabla\dot{\varphi}$-independent)

$$\frac{\partial}{\partial t}\left\{ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\alpha} - \frac{\partial}{\partial x^k} \sum_{k-1}^{n} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \dot{\varphi}^\alpha}{\partial x^k}\right)} \right\} - \left\{ \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \frac{\partial}{\partial x^k} \sum_{k-1}^{n} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^k}\right)} \right\} = 0$$

More compactly

$$\frac{\partial}{\partial t}\frac{\delta \mathcal{L}}{\delta \dot{\varphi}^\alpha} - \frac{\delta \mathcal{L}}{\delta \varphi^\alpha} = 0 \tag{46}$$

where

$$\frac{\delta \mathcal{L}}{\delta \varphi^\alpha} \equiv \left\{ \frac{\partial}{\partial \varphi^\alpha} - \frac{\partial}{\partial x^k} \sum_{k-1}^{n} \frac{\partial}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^k}\right)} \right\} \mathcal{L} \tag{47}$$

serves to define the so-called "variational derivative" of $\mathcal{L}$ with respect to φ^α. Equations (46), which exist in as many variants as there are ways to assign specific meaning to the parameter t, manage—"by contrivance," as it were—to resemble the Lagrange equations of particle mechanics, and put us in position to write out the generic Hamiltonian field theory $\mathcal{H}^t$ by proceeding in formal mimicry of the methods of particle mechanics.

Our former definition of the momentum fields π_α which are (within $\mathcal{H}^t$) conjugate to the φ^α-fields can now be notated

$$\pi_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}^\alpha} \tag{48}$$

Evidently $[\varphi^\alpha][\pi_\alpha] = [\mathcal{L}][\,\textit{physical dimension of the parameter } t\,]$, which when t has the nature of a "time"—but not otherwise—entails

$$[\varphi^\alpha][\pi_\alpha] = \text{ action density} \tag{49}$$

We note also that—whatever the degree of "directly physical significance" that attaches to the φ-fields (and that varies from application to application)—we

should in no case assign "direct physical significance" to the associated π-fields, for they respond non-invariantly to gauge transformations:

$$\mathcal{L} \longrightarrow \mathcal{L} + \text{gauge term} \qquad \text{induces} \qquad \pi_\alpha \longrightarrow \pi_\alpha + \text{gauge term}$$

If the so-called "Hessian" (the determinant of the matrix $\|\partial^2\mathcal{L}/\partial\dot{\varphi}^\alpha\partial\dot{\varphi}^\beta\|$) does not vanish, then it becomes possible-in-principle—by functional inversion of the system of equations

$$\pi_\alpha = \frac{\delta}{\delta\dot{\varphi}^\alpha}\mathcal{L}(\dot{\varphi}, \nabla\varphi; \varphi; t, x) = \pi_\alpha(\dot{\varphi}, \nabla\varphi; \varphi; t, x)$$

—to write $$\dot{\varphi}^\alpha = \dot{\varphi}^\alpha(\pi, \nabla\varphi; \varphi; t, x)$$

and therefore to construct that particular "Legendre transform" of $\mathcal{L}$

$$\begin{aligned}\mathcal{H} &= \pi_\alpha\dot{\varphi}^\alpha - \mathcal{L}(\dot{\varphi}, \nabla\varphi; \varphi; t, x)\Big|_{\dot{\varphi}^\alpha \longrightarrow \dot{\varphi}^\alpha(\pi, \nabla\varphi; \varphi; t, x)} \\ &= \mathcal{H}(\pi, \nabla\varphi; \varphi; t, x)\end{aligned} \tag{50}$$

which (within the $\mathcal{H}^t$ formalism) plays the role of a "Hamiltonian density." The associated "Hamiltonian" would be constructed

$$H = \int \mathcal{H} dx^1 \dots dx^n$$

Clearly, $[\mathcal{H}] = [\mathcal{L}] =$ *energy density* and $[H] =$ *energy*. It is notable that $\mathcal{H}$, as displayed in (50), is devoid of the $\nabla\pi$-dependence one might, on formal grounds, otherwise have expected. So far as concerns its $\nabla\varphi$-dependence, we compute

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} &= \underbrace{\pi_\beta\frac{\partial\dot{\varphi}^\beta}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} - \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^\beta}\frac{\partial\dot{\varphi}^\beta}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)}}_{=\,0 \text{ by the definition of } \pi_\beta} - \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} \\ &= -\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)}\end{aligned} \tag{51}$$

of which we will have immediate need. For by a similar calculation

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\varphi^\alpha} &= \underbrace{\pi_\beta\frac{\partial\dot{\varphi}^\beta}{\partial\varphi^\alpha} - \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^\beta}\frac{\partial\dot{\varphi}^\beta}{\partial\varphi^\alpha}}_{0} - \frac{\partial\mathcal{L}}{\partial\varphi^\alpha} \\ &= -\frac{\partial}{\partial t}\pi_\alpha - \sum_{k=1}^{n}\frac{\partial}{\partial x^k}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} \quad \text{by the field equations}\end{aligned}$$

and when we draw upon (51) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\pi_\alpha &= -\left\{\frac{\partial \mathcal{H}}{\partial \varphi^\alpha} - \sum_{k=1}^{n}\frac{\partial}{\partial x^k}\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial \varphi^\alpha}{\partial x^k}\right)}\right\} \\ &= -\frac{\delta \mathcal{H}}{\delta \varphi^\alpha}\end{aligned} \tag{52}$$

Finally—by a simpler variant of the same line of argument—we have

$$\frac{\partial \mathcal{H}}{\partial \pi_\alpha} = \dot{\varphi}^\alpha + \underbrace{\pi_\beta \frac{\partial \dot{\varphi}^\beta}{\partial \pi_\alpha} - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\beta}\frac{\partial \dot{\varphi}^\beta}{\partial \pi_\alpha}}_{0} = \dot{\varphi}^\alpha$$

which, owing to the fact that (as has already been remarked) $\mathcal{H}$ is actually $\nabla\pi$-dependent, can be notated

$$\frac{\partial}{\partial t}\varphi^\alpha = +\frac{\delta \mathcal{H}}{\delta \pi_\alpha} \tag{53}$$

Pulling these results together, we have

$$\left.\begin{aligned}\dot{\varphi}^\alpha &= +\frac{\delta \mathcal{H}}{\delta \pi_\alpha} \\ \dot{\pi}_\alpha &= -\frac{\delta \mathcal{H}}{\delta \varphi^\alpha}\end{aligned}\right\} \tag{54}$$

which are the field equations in "canonical Hamiltonian form." The derivation of (54) has been designed to resemble maximally its counterpart in particle mechanics.[30] We recognize the cancellations encountered along the way to be a characteristic signature of the Legendre transform, seen in all of its diverse applications.

One might plausibly suppose, in view of the structure of (46) and of (54), that we are off and running; that we are now in position to work our way through (say) Whittaker or Goldstein, painlessly translating the concepts and formulæ basic to the analytical dynamics of particles—one after another, as we come to them—into the language of field theory. For example, we might find it natural in imitation of

$$[A, B] \equiv \sum_{k=1}^{n}\left\{\frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q^k}\frac{\partial A}{\partial p_k}\right\}$$

[30] See, for example, p. 193 of Classical Mechanics (1983). An alternative line of argument, which proceeds not from properties of $\mathcal{H}$ but from properties of the Hamiltonian $H = \int \mathcal{H} dx$ and uses "integration by parts" to motivate the introduction of the variational derivative, see Chapter IV, pp. 19–20 of Classical Theory of Fields (1965).

to introduce a "field-theoretic Poisson bracket" by

$$[\mathcal{A},\mathcal{B}] \equiv \sum_{\varkappa=1}^{N}\left\{\frac{\delta\mathcal{A}}{\delta\varphi^{\varkappa}}\frac{\delta\mathcal{B}}{\delta\pi_{\varkappa}} - \frac{\delta\mathcal{B}}{\delta\varphi^{\varkappa}}\frac{\delta\mathcal{A}}{\delta\pi_{\varkappa}}\right\} \tag{55}$$

and to notice that the canonical equations (54) can in this notation be written

$$\left.\begin{aligned}\dot{\varphi}^{\alpha} &= [\varphi^{\alpha},\mathcal{H}]\\ \dot{\pi}_{\alpha} &= [\pi_{\alpha},\mathcal{H}]\end{aligned}\right\} \tag{56}$$

One appears at this point to possess the seed of a notion of "$\mathcal{H}$-generated flow in phase space"

$$\varphi^{\alpha}(t) \longrightarrow \varphi^{\alpha}(t+dt) = \varphi^{\alpha}(t) + dt\cdot[\varphi^{\alpha},\mathcal{H}]$$

and to stand on the brink of a field-theoretic analog of the theory of canonical transformations. But all is not quite so simple. Surprises lurk... for reasons which have partly to do with the circumstance that $\mathcal{H}(\pi,\nabla\varphi;\varphi;t,x)$ possesses —in its $\nabla\varphi$-dependence—a structural element which is absent from $H(p,q)$. I turn now to a discussion intended to identify more clearly some of the points at issue.

In the Hamiltonian mechanics of particles, the familiar constructions

$$\begin{aligned}\text{coordinate} &= q\\ \text{conjugate momentum} &= p\\ \text{energy} &= \tfrac{1}{2m}p^2 + U(x)\\ \text{angular momentum} &= xp_y - yp_x\\ &\vdots\end{aligned}$$

make it natural to assign the name "observable" to functions of the type

$$A = A(p,q;t) = A(p_1,p_2,\ldots,p_n,q^1,q^2,\ldots,q^n;t)$$

and to notice that, in consequence of the canonical equations of motion,

$$\dot{A} = [A,H] + \frac{\partial}{\partial t}A \tag{57}$$

By natural extension (taking care to cast our net wide enough to include $\mathcal{H}$ itself; see again (50)), we assign the name "observable density" to constructions of the type

$$\mathcal{A} = \mathcal{A}(\nabla\pi,\nabla\varphi,\pi,\varphi,x;t)$$

and look to the evaluation of

$$\dot{\mathcal{A}} = \frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_{\alpha}}{\partial x^{k}}\right)}\frac{\partial}{\partial t}\frac{\partial\pi_{\alpha}}{\partial x^{k}} + \frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\varphi^{\alpha}}{\partial x^{k}}\right)}\frac{\partial}{\partial t}\frac{\partial\varphi^{\alpha}}{\partial x^{k}} + \frac{\partial\mathcal{A}}{\partial\pi_{\alpha}}\dot{\pi}_{\alpha} + \frac{\partial\mathcal{A}}{\partial\varphi^{\alpha}}\dot{\varphi}^{\alpha} + \frac{\partial\mathcal{A}}{\partial t}$$

The trick here is to notice that

$$\begin{aligned}\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\frac{\partial}{\partial t}\frac{\partial\pi_\alpha}{\partial x^k} &= \frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\frac{\partial}{\partial x^k}\dot{\pi}_\alpha \\ &= \frac{\partial}{\partial x^k}\left\{\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\dot{\pi}_\alpha\right\} - \dot{\pi}_\alpha\frac{\partial}{\partial x^k}\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\end{aligned}$$

and that the corresponding φ-term yields to similar manipulation. We therefore have

$$\begin{aligned}\dot{\mathcal{A}} = &\left\{\frac{\partial\mathcal{A}}{\partial\varphi^\alpha} - \frac{\partial}{\partial x^k}\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)}\right\}\dot{\varphi}^\alpha + \left\{\frac{\partial\mathcal{A}}{\partial\pi_\alpha} - \frac{\partial}{\partial x^k}\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\right\}\dot{\pi}_\alpha \\ &+ \frac{\partial}{\partial x^k}\left\{\frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)}\dot{\varphi}^\alpha + \frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\dot{\pi}_\alpha\right\} + \frac{\partial\mathcal{A}}{\partial t}\end{aligned}$$

and if we draw upon the canonical equations (54), recall the definition (47) of the variational derivative, and note that

$$\frac{\delta\mathcal{A}}{\delta\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} = \frac{\partial\mathcal{A}}{\partial\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)} \quad \text{because } \mathcal{A} \text{ is } \nabla\nabla\mathcal{A}\text{-independent}$$

we obtain

$$\begin{aligned}\dot{\mathcal{A}} = &\left\{\frac{\delta\mathcal{A}}{\delta\varphi^\alpha}\frac{\delta\mathcal{H}}{\delta\pi_\alpha} - \frac{\delta\mathcal{H}}{\delta\varphi^\alpha}\frac{\delta\mathcal{A}}{\delta\pi_\alpha}\right\} + \frac{\partial\mathcal{A}}{\partial t} \\ &+ \frac{\partial}{\partial x^k}\left\{\frac{\delta\mathcal{A}}{\delta\left(\frac{\partial\varphi^\alpha}{\partial x^k}\right)}\frac{\delta\mathcal{H}}{\delta\pi_\alpha} - \frac{\delta\mathcal{H}}{\delta\varphi^\alpha}\frac{\delta\mathcal{A}}{\delta\left(\frac{\partial\pi_\alpha}{\partial x^k}\right)}\right\}\end{aligned} \tag{58}$$

At this point field theory and particle mechanics appear to have diverged, for the dangling term in (58)—which is present except in special cases of the type $\mathcal{A}(\pi,\varphi)$—has no counterpart in (57). If, however, we allow ourselves to write

$$\text{“observable”} = \int \text{“observable density”} dx^1\cdots dx^n$$

then we can proceed from (58) to the conclusion

$$\begin{aligned}\dot{A} &= \int\dot{\mathcal{A}}dx \\ &= \int\left\{[\mathcal{A},\mathcal{H}] + \frac{\partial\mathcal{A}}{\partial t}\right\}dx + \text{surface term} \\ &\downarrow \\ &= \int\left\{[\mathcal{A},\mathcal{H}] + \frac{\partial\mathcal{A}}{\partial t}\right\}dx \quad \text{when the “surface term” vanishes}\end{aligned} \tag{59}$$

At (55) we assigned meaning to the “Poisson bracket of a pair of observable *densities*.” The result just achieved invites us to write

$$[A,B] \equiv \int[\mathcal{A},\mathcal{B}]dx \tag{60}$$

and thus to assign meaning to the "Poisson bracket of a pair of *observables*." The two concepts are clearly related, yet clearly distinct. If we return with this notation to (59) we find ourselves writing an equation which is identical in appearance to (57), but which carries now a field-theoretic meaning. In particular, if hypotheses sufficient to force the "surface term" to vanish are in place, and if $\partial\mathcal{A}/\partial t = 0$ (i.e., if the observable A is devoid of any *explicit* t-dependence) then

$$\dot{A} = [A, H] \tag{61}$$

From (61) we can recover (56) as special cases, and are led to the conclusion that

$$[A, H] = 0 \quad \Longrightarrow \quad A = \int \mathcal{A} dx \text{ is a } \textit{constant of the field-motion}$$

Evidently $[A, H] = 0$ provides a global formulation of the local statement $[\mathcal{A}, \mathcal{H}] = 0$.

It becomes instructive at this point to revisit our former discussion of Noether's Theorem, which recent remarks[31] will have recalled to the minds of attentive readers. Noether was led from the specification of certain "maps" to the construction at (29) of certain expressions $J_r^k(\varphi, \partial\varphi, x)$—constructions which within the Hamiltonian formalism $\mathcal{H}^i$ acquire the status of "observable densities" of a particular design:

$$\left.\begin{aligned} \mathcal{J}_r^0 &= \qquad \pi_\alpha \left\{ \Phi_r^\alpha - \big(\dot{\varphi}^\alpha \mathcal{X}_r^0 + \sum \frac{\partial \varphi^\alpha}{\partial x^k} \mathcal{X}_r^k\big) \right\} + \big(\pi_\alpha \dot{\varphi}^\alpha - \mathcal{H}\big)\mathcal{X}_r^0 \\ \mathcal{J}_r^i &= \frac{\partial \mathcal{L}}{\partial\big(\frac{\partial \varphi^\alpha}{\partial x^i}\big)} \left\{ \Phi_r^\alpha - \big(\dot{\varphi}^\alpha \mathcal{X}_r^0 + \sum \frac{\partial \varphi^\alpha}{\partial x^k} \mathcal{X}_r^k\big) \right\} + \big(\pi_\alpha \dot{\varphi}^\alpha - \mathcal{H}\big)\mathcal{X}_r^i \end{aligned}\right\} \tag{62}$$

Here $\mathcal{X}_r^k(t, x)$ and $\Phi_r^\alpha(\varphi; t, x)$ are considered to have been prescribed, and $\dot{\varphi}^\alpha$ is to be read as it was during the assembly at (50) of $\mathcal{H}$; i.e., as a reference to $\dot{\varphi}_r^\alpha(\pi, \nabla\varphi; \varphi; t, x)$. Local conservation laws rooted in Noether's Theorem—rooted, that is to say, in the Lagrangian formalism—have (recall (32)) the characteristic form

$$\partial_k J_r^k = 0$$

Such statements are, as I have already emphasized, to be read as *implications of the field equations*; i.e., of the Lagrange equations of motion, of which they express a revealed symmetry property. But, so far as I am aware, there exists no generally-applicable technique for explicitly *demonstrating* that

$$\text{field equations} \quad \Longrightarrow \quad \partial_k J_r^k = 0$$

Such problems are tackeled case-by-case, by *ad hoc* methods special to the instance, as witnessed in the discussion subsequent to (34). But consider: Noetherian conservation laws admit (see again (40)) of *global* formulation

$$\dot{J}_r = 0 \quad \text{with} \quad J_r = \int J_r^0 dx$$

[31] Compare p. 32.

And this is a statement which translates directly into language natural to the Hamiltonian formalism

$$\dot{J}_r = 0 \quad \text{with} \quad J_r = \int \mathcal{J}^0_r dx$$

where it leads to a local statement

$$[\, \mathcal{J}^0_r, \mathcal{H}\,] = 0$$

which is in every computational respect quite distinct from its Lagrangian counterpart, $\partial_k J^k_r = 0$. Interestingly, what was a bothersome gap within the Lagrangian formalism is a question rendered moot in the Hamiltonian formalism; a generally-applicable technique for explicitly demonstrating *how it comes about* that

$$\text{canonical field equations} \quad \Longrightarrow \quad [\, \mathcal{J}^0_r, \mathcal{H}\,] = 0$$

is written onto the very face of the statement. Interesting also is the fact that, while statements of the form $\partial_k J^k = 0$ just sit there as unitary thoughts within the Lagrangian formalism, they admit of as many (generally distinct and complementary) modes of Hamiltonian interpretiation as there are variants of the Hamiltonian formalism. I draw attention finally to the fact that the densities $\mathcal{J}^0_k(\nabla\varphi, \varphi, x; t)$ described by (62) are structurally quite particular; evidently one cannot expect to construct, in any natural way, a "Noetherian interpretation" of the conservation law $[\mathcal{A}, \mathcal{H}\,] = 0$ if $\mathcal{A}$ fails to exhibit the required "structural particularities." This is the point I had in mind at footnote [16].

It had become apparent by the time we reached (61) that—the evidence of (54) notwithstanding—a properly drawn abstract of the relationship between Hamiltonian particle mechanics and the Hamiltonian theory of fields reads not

$$H(p, q) \quad \longleftrightarrow \quad \mathcal{H}$$

but

$$H(p, q) \quad \longleftrightarrow \quad H = \int \mathcal{H} dx$$

—as would, in fact, have been evident from the outset had we proceeded by the lattice-refinement technique. But consider: field-theoretic objects of type $H, J_r, A, B, \ldots$ are by nature *functions of functions* [32]—they are, in short, "functionals"—while the notion of a "Poisson bracket" is, in all of its diverse manifestions, rooted in the concept of *differentiation*. The Poisson bracket guards the entry portal to the higher reaches of Hamiltonian mechanics; pretty clearly, if we, as field theorists, are ever to penetrate those higher reaches—are ever, for example, to understand the deeper meaning of (61)—we must be in possession of a "functional calculus." That, therefore, is a topic to which I promise to return. But for the moment it seems to me advisable to take temporary leave of theory-building in order to explore what the results already in hand have to say about illustrative concrete cases.

[32] They are, more precisely, number-valued functions of *sets* of functions $\{\pi, \varphi\}$ *and their "spatial" first partials* $\{\nabla\pi, \nabla\varphi\}$.

Examples of the Hamiltonian method at work. Look first to the single-field system

$$\mathcal{L} = \tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2 - F(\varphi)$$

first encountered on p. 37. There are two independent variables, therefore two distinct variants ($\mathcal{H}^t$ and $\mathcal{H}^x$) of the Hamiltonian formalism. Working first within the former, we write

$$\mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}\varphi_x^2 - F(\varphi) \quad \text{with} \quad \dot{\varphi} \equiv \varphi_t$$

and introduce

$$\pi = \partial\mathcal{L}/\partial\dot{\varphi} = \dot{\varphi}$$

The function $\dot{\varphi}(\pi, \varphi_x, \varphi)$ is in this case very simple: $\dot{\varphi} = \pi$. The Hamiltonian density is given therefore by

$$\begin{aligned}\mathcal{H} &= \left[\pi\dot{\varphi} - \mathcal{L}(\varphi, \dot{\varphi}, \varphi_x)\right]_{\dot{\varphi}\to\pi} \\ &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\varphi_x^2 + F(\varphi)\end{aligned}$$

The canonical equations of motion (54) read

$$\begin{aligned}\dot{\varphi} &= +\left\{\frac{\partial}{\partial\pi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\pi_x}\right\}\mathcal{H} = \pi \\ \dot{\pi} &= -\left\{\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\varphi_x}\right\}\mathcal{H} = -F'(\varphi) + \varphi_{xx}\end{aligned}$$

from which it is very easy to recover the Lagrangian field equation

$$\varphi_{tt} - \varphi_{xx} + F'(\varphi) = 0$$

The Noetherian analysis of p. 28, applied to the specific system now at hand, shows it to be an implication of the assumed t-independence of $\mathcal{L}$ that

$$\partial_t S^t{}_t + \partial_x S^x{}_t = 0$$

while

$$\partial_t S^t{}_x + \partial_x S^x{}_x = 0$$

follows similarly from the assumed x-independence. Explicit descriptions of the quantities

$$\begin{pmatrix} S^t{}_t & S^t{}_x \\ S^x{}_t & S^x{}_x \end{pmatrix} \equiv \begin{pmatrix} S^0{}_0 & S^0{}_1 \\ S^1{}_0 & S^1{}_1 \end{pmatrix}$$

can be read off from (34), but since in the Hamiltonian formalism $\mathcal{H}^t$ we have interest only in $S^t{}_t$ and $S^t{}_x$ we restrict our explicit attention to those; we find

$$\begin{aligned}S^t{}_t &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\dot{\varphi} - \mathcal{L} \\ S^t{}_x &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\varphi_x\end{aligned}$$

from which (using $\dot\varphi = \pi$ to eliminate all reference to $\dot\varphi$) we obtain

$$\begin{aligned}\mathcal{S}_t &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\varphi_x^2 + F(\varphi)\\ \mathcal{S}_x &= \pi\varphi_x\end{aligned}$$

Since $\mathcal{S}_t$ *is precisely the Hamiltonian density* $\mathcal{H}$, it is trivially the case that $[\,\mathcal{S}_t, \mathcal{H}\,] = 0$, and therefore trivial also that

$$S_t = \int \mathcal{S}_t dx \quad \text{is conserved:} \quad \dot S_t = \int \underbrace{[\,\mathcal{S}_t, \mathcal{H}\,]}_{0} dx = 0$$

By calculation we find, on the other hand, that

$$\begin{aligned}[\,\mathcal{S}_x, \mathcal{H}\,] &= -\pi_x\pi - \varphi_x\big(F'(\varphi) - \varphi_{xx}\big)\\ &= \tfrac{\partial}{\partial x}\mathcal{W} \quad \text{with} \quad \mathcal{W} \equiv \tfrac{1}{2}\varphi_x^2 - \tfrac{1}{2}\pi^2 - F(\varphi)\end{aligned}$$

does *not* vanish, but *has the structure of a divergence*. Therefore

$$\dot S_x = \int \tfrac{\partial}{\partial x}\mathcal{W}\, dx = \text{boundary terms}$$

from which it follows that

$$S_x = \int \mathcal{S}_x dx \quad \text{is conserved if the boundary terms vanish}$$

Since, as we have seen, S_x admits of physical interpretation as the *total linear momentum* of the system, we are not at all surprised to be reminded that "conservation of linear momentum requires that the system be isolated" (no boundary effects).

We follow those same ideas now down a less well-trodden path. Relativistic considerations (which we are not yet in position to entertain, though they are of no great profundity; we note simply that our $\mathcal{L}$ contains $\mathcal{L} = \varphi_{tt} - \varphi_{xx}$ as a special case, and that this is the system which historically served as the cradle of Special Relativity) inspire an interest in the Lorentz-transform properties of the system $\mathcal{L}$. In 2-dimensional spacetime the Lorentz group is a one-parameter group, the elements of which can be described

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \longrightarrow \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} = e^{\omega\mathbb{G}} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \quad \text{with} \quad \mathbb{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To describe an infinitesimal Lorentz transformation we therefore write

$$\begin{aligned}\delta_\omega x^0 &= \mathcal{X}^0\delta\omega \quad \text{with} \quad \mathcal{X}^0 = x^1\\ \delta_\omega x^1 &= \mathcal{X}^1\delta\omega \quad \text{with} \quad \mathcal{X}^1 = x^0\end{aligned}$$

and, to express the presumption that φ transforms as a *scalar* field, set $\Phi = 0$. Returning now to (29) (and reverting to our recent notational conventions: $t \longleftarrow x^0$, $x \longleftarrow x^1$) we obtain the Noetherian current with components

$$\begin{aligned} K^t &= +\varphi_t(\varphi_t x + \varphi_x t) - (\tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2 - F)x \\ K^x &= -\varphi_x(\varphi_t x + \varphi_x t) - (\tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2 - F)t \end{aligned}$$

which after simplifications can be written

$$\begin{aligned} K^t &= +\varphi_t\varphi_x t + (\tfrac{1}{2}\varphi_t^2 + \tfrac{1}{2}\varphi_x^2 + F)x \\ K^x &= -\varphi_t\varphi_x x - (\tfrac{1}{2}\varphi_t^2 + \tfrac{1}{2}\varphi_x^2 - F)t \end{aligned}$$

If one writes out $\partial_t K^t + \partial_x K^x$ and draws upon the Lagrangian field equation one discovers without difficulty that in fact

$$\partial_t K^t + \partial_x K^x = 0$$

But from a Hamiltonian point of view—more precisely, from the viewpoint of the $\mathcal{H}^t$ formalism—the object of interest is the observable density

$$\mathcal{K} \equiv K^t\big|_{\varphi_t \longrightarrow \pi} = \pi\varphi_x t + \mathcal{H}x$$

Noting that $\mathcal{K}$ displays some *explicit* t-dependence, we compute

$$\begin{aligned} [\,\mathcal{K}, \mathcal{H}\,] + \frac{\partial \mathcal{K}}{\partial t} &= \big\{F'x - (\pi t)_x - (\varphi_x x)_x\big\}\big\{\pi\big\} - \big\{F' - \varphi_{xx}\big\}\big\{\varphi_x t + \pi x\big\} + \pi\varphi_x \\ &= t \cdot \tfrac{\partial}{\partial x}\mathcal{W} \quad \text{with} \quad \mathcal{W} \equiv \tfrac{1}{2}\varphi_x^2 - \tfrac{1}{2}\pi^2 - F(\varphi) \quad \text{as before} \end{aligned}$$

Therefore

$$\dot{K} = \int \Big\{[\,\mathcal{K}, \mathcal{H}\,] + \tfrac{\partial}{\partial t}\mathcal{K}\Big\}dx = t \cdot \int \tfrac{\partial}{\partial x}\mathcal{W}dx = \text{boundary terms}$$

from which it follows that

$$K = \int \mathcal{K}dx \quad \text{is conserved if the boundary terms vanish}$$

So distinguished is the ancestry of K—it is rooted in, and a symptom of, the Lorentz covariance of our system—that some urgency attaches to the matter of its physical interpretation. The following observations are intended not to resolve that question, but only to indicate the direction in which the resolution surely lies. We begin by noting that $\int(\mathcal{H}x)dx$ is the first moment of the energy distribution. Equivalently, the first moment of the mass distribution. It becomes on this basis fairly natural to consider that

$$X(t) = \int(\mathcal{H}x)dx \quad \text{locates the "center of mass" of the field}$$

In this notation $K = X(0)$, and we have

$$X(t) = X(0) + Vt$$

with $V = -\int(\pi\varphi_x)dx = -\int(\pi\varphi)_x dx + \int(\pi_x\varphi)dx$, where $\int(\pi\varphi)_x dx$ has in fact the nature of a boundary term, and might therefore be dropped. More to the point: if we write $V = \int \mathcal{V}dx$ then by quick calculation we find

$$[\mathcal{V}, \mathcal{H}] = -\tfrac{\partial}{\partial x}\mathcal{V} - \pi_x F(\varphi)$$

from which it follows that V is itself a constant of the field motion, except for effects attributable to $F(\varphi)$. In the absence of such effects, we can, by these interpretations, consider $K = 0$ to be telling us that—however complicated the motion of the field itself may be, its center of mass drifts with constant velocity. In the mechanics of N-particle systems, Galilean covariance can be exploited to similar effect.[33]

Now we take off our Hamiltonian hat $\mathcal{H}^t$, put on the hat $\mathcal{H}^x$, and look to the physics of the same system as before. We agree to retain the convention that the occurance of an "overdot" $\dot{}$ signifies *differentiation with respect to "the parameter,"* and accept that—since "the parameter" is now not t but x—all allusions to "motion" have acquired suddenly a novel meaning. We write

$$\mathcal{L} = \tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\dot{\varphi}^2 - F(\varphi) \quad \text{with} \quad \dot{\varphi} \equiv \varphi_x$$

and assign new meaning to the conjugate momentum field[34]

$$\pi = \partial\mathcal{L}/\partial\dot{\varphi} = -\dot{\varphi}$$

The Hamiltonian density is given in $\mathcal{H}^x$ therefore by

$$\begin{aligned}\mathcal{H} &= \left[\pi\dot{\varphi} - \mathcal{L}(\varphi, \varphi_t, \dot{\varphi})\right]_{\dot{\varphi}\to-\pi} \\ &= -\tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\pi^2 + F(\varphi)\end{aligned}$$

It is, as it was in $\mathcal{H}^t$, a Legendre transform of $\mathcal{L}$, but a *different* Legendre transform. The canonical equations of motion now read

$$\begin{aligned}\dot{\varphi} &= +\left\{\frac{\partial}{\partial\pi} - \frac{\partial}{\partial t}\frac{\partial}{\partial\pi_t}\right\}\mathcal{H} = -\pi \\ \dot{\pi} &= -\left\{\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial t}\frac{\partial}{\partial\varphi_t}\right\}\mathcal{H} = -F'(\varphi) - \varphi_{tt}\end{aligned}$$

[33] See Classical Mechanics (1983), pp. 170 & 253 and especially—because it is much more "field-theoretic" in spirit—the discussion which appears in Classical Electrodynamics (1980) at pp. 323–329.

[34] In order to keep simple things simple, I am asking my reader simply to *remember* that π and similar constructs have now new meanings; to distinguish π^t from π^x seems excessively pedantic, and lends off-putting clutter to the symbols which denote the partial derivatives of those objects.

from which it is as easy as it was before to recover the Lagrangian field equation

$$\varphi_{tt} - \varphi_{xx} + F'(\varphi) = 0$$

We have observed that—and why—the $\mathcal{H}^t$ formalism exhibits a special interest in the t-row of the stress-energy tensor, and are not surprised to discover that the $\mathcal{H}^x$ formalism is similarly enamoured of the x-row. We have

$$\begin{aligned} S^x{}_t &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \varphi_t \\ S^x{}_x &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} \end{aligned}$$

from which we obtain

$$\begin{aligned} \mathcal{S}_t &= \pi \varphi_t \\ \mathcal{S}_x &= -\tfrac{1}{2}\varphi_t^2 - \tfrac{1}{2}\pi^2 + F(\varphi) \end{aligned}$$

Reminding ourselves that the variational derivatives $\frac{\delta}{\delta \varphi}$ and $\frac{\delta}{\delta \pi}$ have now (see again the most recent version of the canonical equations of motion) a new meaning, and that so also does the Poisson bracket, we compute

$$[\,\mathcal{S}_t, \mathcal{H}\,] = \tfrac{\partial}{\partial t}\mathcal{U} \quad \text{with} \quad \mathcal{U} \equiv -\tfrac{1}{2}\varphi_t^2 + \tfrac{1}{2}\pi^2 - F(\varphi)$$

We introduce $S_t = \int \mathcal{S}_t \, dt$ and from

$$\dot{S}_t = \int [\,\mathcal{S}_t, \mathcal{H}\,]\, dt = \text{temporal boundary terms}$$

conclude that S is "conserved"—a constant of the "motion"—if the "temporal boundary terms" vanish. A simpler argument (one has only to notice that $\mathcal{S}_x = \mathcal{H}$, from which $\dot{S}_x = \int [\,\mathcal{S}_x, \mathcal{H}\,]\, dt = 0$ follows trivially) establishes the unconditonal conservation of $S_x = \int \mathcal{S}_x \, dt$.

The $\mathcal{H}^x$ formalism has led us, with an inevitability born of its familiar internal logic, to the perception of a population of conservation laws of this unfamiliar general type:

> Sit in Portland for an eternity, collecting data sufficient to permit the evaluation of (say) $S_x = \int \mathcal{S}_x \, dt$. The result you ultimately obtain is the same as you would have obtained had you instead elected to sit in Pocatello. Or Paris, or Prague.

Such statements are "bizarre" for precisely the reason, and to precisely the extent, that "motion" within $\mathcal{H}^x$ is bizarre: it is x-parameterized, and therefore runs counter to the mechanical experience which has made unwitting "time-Chauvinists" of us all—relativistically untenable though we recognize such a position to be. In particle mechanics our "time-Chauvinism" is, I think, pretty much forced upon us; we may, if we wish, tinker with the metrization of time ($t \longrightarrow \theta = \theta(t)$), but the subject-matter provides no independent

variable fundamentally distinct from the time parameter, and no plausible alternative to such statements as this: "A particle can easily be at the same point at two times, but cannot easily be at two points at the same time." But now we are doing field theory, which provides an abundance of alternatives to t-parameterization, and which treats systems that are *typically* "at two points at the same time." Perhaps, therefore, we should relax the tenacity with which we cling to some entrenched habits of thought. Why, after all, do we collect conservation theorems? For a variety of reasons, of which some are more narrowly technical than others, but principally because *conservation laws nourish physical intuition*; they permit us to see unity in the contingent details

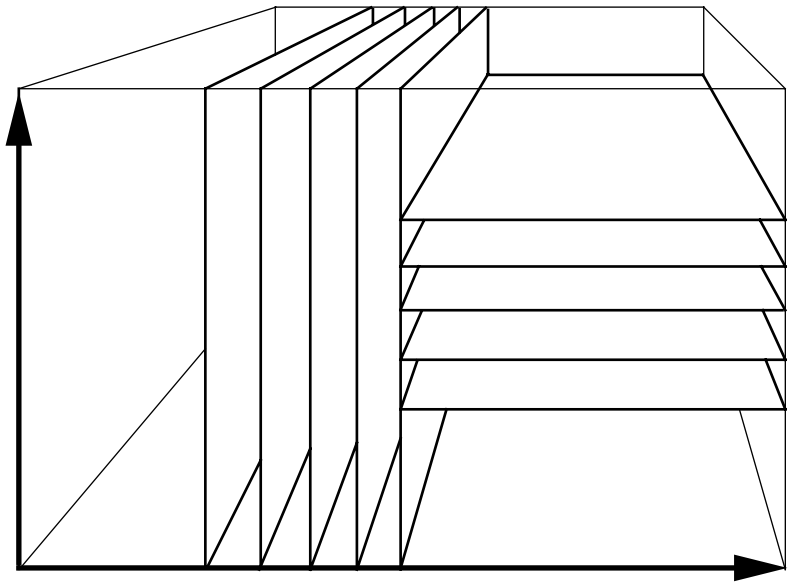


FIGURE 8: *The "standard" Hamiltonian formalism $\mathcal{H}^t$ supplies conservation theorems which are written onto consecutive t-slices, as illustrated on the right half of the figure. The formalism $\mathcal{H}^x$ supplies, on the other hand, theorems which incorporate data written onto consecutive x-slices. It is argued in the text that both contribute usefully to a fuller understanding of the physical system φ, which in itself just "sits there, waiting to be probed" in as many ways as we can think of.*

of particular cases, in the confusing complexity typical of particular solutions of the equations of motion (in field theory: the field equations); they permit us even to say useful things when no solutions are known. It would seem in this light extravagant to discard such "conservation theorems" as $\mathcal{H}^x$ stands ready to provide, simply on the grounds that they come to us nameless, and require of us that we exercise some unfamiliar modes of thought. Nor, in the end, would

such a "policy of abandonment" even be tenable. For if

$$t \longrightarrow T = T(t,x)$$
$$x \longrightarrow X = X(t,x)$$

then the conservation theorems which issue from $\mathcal{H}^T$ will , in general, *conflate* those which issue from $\mathcal{H}^t$ and $\mathcal{H}^x$. As so also, of course, will those which issue from $\mathcal{H}^X$. Directly related to the preceeding remark is this final curious point: We have proceeded from a comparison of the output (so far as relates to conservation theorems) of the $\mathcal{H}^x$ formalism with that of the more familiar $\mathcal{H}^t$ formalism. But there is, in fact, nothing peculiarly "Hamiltonian" about our conclusions, for they were latent already in the Lagrangian formalism, where the conservation law

$$\partial_i J^i = 0$$

is standardly elaborated

$$\partial_0 J^0 + \text{divergence term} = 0$$
$$\text{divergence term} \equiv \quad \bullet \quad + \partial_1 J^1 + \partial_2 J^2 + \cdots + \partial_n J^n$$

but *can*—equally well, if non-standardly—be elaborated

$$\partial_1 J^1 + \text{divergence term} = 0$$
$$\text{divergence term} \equiv \partial_0 J^0 + \quad \bullet \quad + \partial_2 J^2 + \cdots + \partial_n J^n$$

To write the former is to acquire automatic interest in the t-independence of

$$\int J^0 dx^1 dx^2 \cdots dx^n$$

while the latter engenders interest in the x^1-independence of

$$\int J^1 \; dt \; dx^2 \cdots dx^n$$

This simple point—brought naturally and forcibly to our attention by the Hamiltonian formalism—is (almost unaccountably) passed by unnoticed in most surveys of Lagrangian field theory.

Look back again now to the two-field system

$$\mathcal{L} = \tfrac{1}{2}(\alpha_t \varphi - \alpha \varphi_t) - \alpha_x \varphi_x$$

which engaged our interest already on p. 38 for the reason that it yields the diffusion equation $\varphi_t = \varphi_{xx}$ as a field equation.[35] In naive preparation for what we intend to be "work within the $\mathcal{H}^t$ formalism," we write

$$\mathcal{L} = \tfrac{1}{2}(\dot{\alpha}\varphi - \alpha\dot{\varphi}) - \alpha_x \varphi_x$$

[35] Recall that we had no initial interest in the auxiliary field α, which was introduced as a formal crutch, and was found to satisfy the "backwards (or time-reversed) diffusion equation."

and introduce the conjugate momenta

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \quad \text{and} \quad \beta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\alpha}}$$

—only to find that we have run smack into a brick wall. For the resulting equations

$$\pi = -\tfrac{1}{2}\alpha$$
$$\beta = +\tfrac{1}{2}\varphi$$

show that the conjugate fields $\{\pi, \beta\}$ are redundant with their companion fields $\{\varphi, \alpha\}$. One cannot by inversion obtain equations of the form

$$\dot{\varphi} = \dot{\varphi}(\text{fields, conjugate fields, non-parametric partials of those})$$
$$\dot{\alpha} = \dot{\alpha}(\text{fields, conjugate fields, non-parametric partials of those})$$

The procedure

$$\mathcal{L} \xrightarrow[\text{Legendre transformation}]{} \mathcal{H}$$

cannot be carried out, for essentially the reason that the Hessian vanishes:

$$\begin{vmatrix} \partial^2\mathcal{L}/\partial\dot{\varphi}\partial\dot{\varphi} & \partial^2\mathcal{L}/\partial\dot{\varphi}\partial\dot{\alpha} \\ \partial^2\mathcal{L}/\partial\dot{\alpha}\partial\dot{\varphi} & \partial^2\mathcal{L}/\partial\dot{\alpha}\partial\dot{\alpha} \end{vmatrix} = 0$$

But consider: if we take

$$\mathcal{H} = -\varphi_x \pi_x$$

to be (within the $\mathcal{H}^t$ formalism) the Hamiltonian density of a *one*-field system, then the canonical equations

$$\dot{\varphi} = +\left\{ \frac{\partial}{\partial \pi} - \frac{\partial}{\partial x}\frac{\partial}{\partial \pi_x} \right\}\mathcal{H} = +\varphi_{xx} \quad : \quad \text{diffusion equation}$$
$$\dot{\pi} = -\left\{ \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial x}\frac{\partial}{\partial \varphi_x} \right\}\mathcal{H} = -\pi_{xx} \quad : \quad \text{backwards diffusion equation}$$

reproduce precisely the equations formerly obtained as field equations from the Lagrangian density $\mathcal{L}$! What we have in hand—since clearly no procedure of the type

$$\mathcal{L} \xleftarrow[\text{Legendre transformation}]{} \mathcal{H}$$

is possible—is a strikingly efficient example of a "free-standing Hamiltonian theory," a theory unsupported by any underlying Lagrangian framework. The absence of such a framework carries with it the implication that we are cut off from (for example) the assistance of Noether's Theorem; when we undertake to construct descriptions of (say) the energy density, or the momentum density—constructions which the equivalent two-field Lagrangian theory supplies straightforwardly—we will be "on our own." But that circumstance we are prepared to accept as an invitation to invention!

There remains yet some juice to be squeezed from the preceeding example, and the last drop is, in a way, the sweetest. By way of preparation, recall that in mechanics of particles it sometimes proves useful—particularly in connection with the theory of canonical transformations[36]—to notice that the canonical equations of motion

$$\dot{q} = +\frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

can be obtained as the "Lagrange equations" from a certain "meta-Lagrangian." For if

$$\boldsymbol{L} \equiv \tfrac{1}{2}(p\dot{q} - \dot{p}q) - H(p,q) = (p\dot{q} - H) + \text{gauge term}$$

then

$$\left\{\frac{d}{dt}\frac{\partial}{\partial \dot{p}} - \frac{\partial}{\partial p}\right\}\boldsymbol{L} = 0 \quad \Longrightarrow \quad \dot{q} = +\frac{\partial H}{\partial p}$$
$$\left\{\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right\}\boldsymbol{L} = 0 \quad \Longrightarrow \quad \dot{p} = -\frac{\partial H}{\partial q}$$

This remark admits straightforwardly of field-theoretic imitation, for if we define

$$\boldsymbol{\mathcal{L}} \equiv \tfrac{1}{2}(\pi\dot{\varphi} - \dot{\pi}\varphi) - \mathcal{H}(\pi,\varphi) \tag{63}$$

we obtain

$$\left.\begin{aligned}\left\{\frac{\partial}{\partial t}\frac{\delta}{\delta \dot{\pi}} - \frac{\delta}{\delta \pi}\right\}\boldsymbol{\mathcal{L}} = 0 \quad &\Longrightarrow \quad \dot{\varphi} = +\frac{\delta \mathcal{H}}{\delta \pi} \\ \left\{\frac{\partial}{\partial t}\frac{\delta}{\delta \dot{\varphi}} - \frac{\delta}{\delta \varphi}\right\}\boldsymbol{\mathcal{L}} = 0 \quad &\Longrightarrow \quad \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \varphi}\end{aligned}\right\} \tag{64}$$

This is a result of very general importance. But look now again to the particular case $\mathcal{H} = -\pi_x\varphi_x$; we have

$$\begin{aligned}\boldsymbol{\mathcal{L}} &= \tfrac{1}{2}(\pi\dot{\varphi} - \dot{\pi}\varphi) + \pi_x\varphi_x \\ &= -\underbrace{\left[\tfrac{1}{2}(\dot{\pi}\varphi - \pi\dot{\varphi}) - \pi_x\varphi_x\right]}_{\mathcal{L} = \frac{1}{2}(\dot{\alpha}\varphi - \alpha\dot{\varphi}) - \alpha_x\varphi_x \text{ to within notational adjustment}}\end{aligned}$$

We've been "talking prose all our lives, without knowing it!" What we initially imagined to be an unwelcome "auxiliary field" was actually the *conjugate* field, and what we took to be the Lagrangian $\mathcal{L}$ was actually the *meta*-Lagrangian, $\boldsymbol{\mathcal{L}}$. When we lamented the impossibility of proceeding

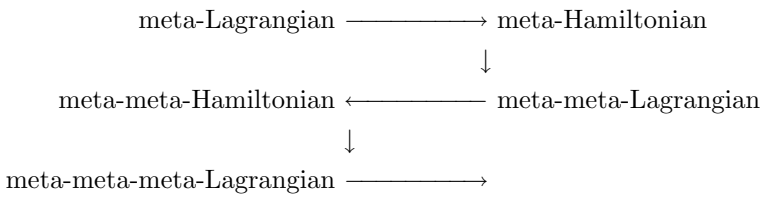
$$\mathcal{L} \xrightarrow[\text{Legendre transformation}]{} \mathcal{H}$$

we were actually commenting on the impossibility of proceeding

$$\boldsymbol{\mathcal{L}} \xrightarrow[\text{Legendre transformation}]{} \boldsymbol{\mathcal{H}}$$

[36] See Classical Mechanics (1983), pp. 217–228.

which is, in fact, cause for rejoicing, for it protects us from the possiblity of infinite regress:



Here I break off this account of the Hamiltonian method, as it applies to the classical theory of fields. My objective has been simply to sketch the lay of the land, and I think that limited objective has been achieved. But clearly, wonders lie just over the horizon.

Classical field theory of a quantum particle. My purpose here will be to illustrate —in what I think readers will agree is a rather surprising context—how the material presently at our command can be used to organize the discussion of some real physics of undeniable importance. In view of that intent, and in order to costume my results in their most familiar dress, I will abandon my formerly somewhat generic mode of expression; I will be careful to insure that all variables are clearly dimensioned, and that all physical parameters and constants are explicitly displayed. Let us agree at the outset that we have equipped ourselves with a "good clock" and an "inertial meter stick;" i.e., that we have coordiniatized spacetime in such a way that the motion of classical free particles can be described $\ddot{x} = 0$. Thus prepared...

To describe the one-dimensional quantum motion of a mass point m in the presence of a potential $U(x)$, we write and undertake to solve—subject to prescribed side conditions and to the normalization condition

$$\int \psi^* \psi dx = 1 \tag{65}$$

—the time-dependent Schrödinger equation

$$-(\hbar^2/2m)\psi_{xx} + U\psi = i\hbar\psi_t \tag{66}$$

The wave function $\psi(x,t)$ is, owing to the presence of the exposed i-factor on the right side of (66), necessarily complex. If, to emphasize that fact, we write

$$\psi = \varphi^1 + i\varphi^2$$

we find that (66) is equivalent to the following coupled *system* of equations

$$\left.\begin{aligned} -(\hbar^2/2m)\varphi^1_{xx} + U\varphi^1 &= -\hbar\varphi^2_t \\ -(\hbar^2/2m)\varphi^2_{xx} + U\varphi^2 &= +\hbar\varphi^1_t \end{aligned}\right\} \tag{67}$$

where evidently it is the "exposed i-factor" that is responsible for the coupling. If, alternatively, we adopt the polar representation

$$\psi = R\exp\left\{\tfrac{i}{\hbar}S\right\}$$

we obtain

$$\left\{-\tfrac{\hbar^2}{2m}\left[R_{xx}+2\tfrac{i}{\hbar}R_xS_x+\tfrac{i}{\hbar}R\left(S_{xx}+\tfrac{i}{\hbar}S_x^2\right)\right]+UR\right\}e^{\frac{i}{\hbar}S}=i\hbar\left(R_t+\tfrac{i}{\hbar}RS_t\right)e^{\frac{i}{\hbar}S}$$

giving—if we abandon the exponential factors, separate the real terms from the imaginary, and make some notational adjustments—

$$\left.\begin{aligned}\tfrac{1}{2m}S_x^2+U+S_t&=\tfrac{\hbar^2}{2m}R^{-1}R_{xx}\\(R^2)_t+(\tfrac{1}{m}S_xR^2)_x&=0\end{aligned}\right\}\tag{68}$$

Or we might, in place of the fields φ^1 and φ^2, elect to work with some linear combination of those fields, writing

$$\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}$$

If, in particular, we elect to set

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & +i\\ 1 & -i\end{pmatrix}$$

we obtain

$$\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\psi\\ \psi^*\end{pmatrix}$$

This little observation lends legitimacy to the frequently very convenient trick whereby one treats ψ and ψ^* as though they were *distinct and independent complex fields*, even though one cannot, in point of fact, change either without inducing change in the other. If we take advantage of this tricky opportunity (as in the future we will consider ourselves free to do casually, without explicit comment), then (66) acquires a co-equal conjugated companion; we have this *uncoupled pair* of Schrödinger equations

$$\left.\begin{aligned}-(\hbar^2/2m)\psi_{xx}+U\psi&=+i\hbar\psi_t\\-(\hbar^2/2m)\psi^*_{xx}+U\psi^*&=-i\hbar\psi^*_t\end{aligned}\right\}\tag{69}$$

It is useful also to note that the Schrödinger equation can, in the case $U=0$, be written

$$\psi_{xx}=D\psi_t\quad\text{with}\quad D\equiv\tfrac{2m}{i\hbar}$$

which has formally the structure of a "diffusion equation (heat equation) with an *imaginary diffusion coefficient*."

The idea now—just crazy enough to be interesting—is to look upon (67), (68) and (69) as providing alternative representations of the *field equations characteristic of a two-component classical (!) field theory*. Semi-arbitrarily electing to work in language provided by the latter representation, we form

$$\left.\begin{aligned} G_1 &\equiv (\hbar^2/2m)\psi_{xx} + i\hbar\psi_t - U\psi \\ G_2 &\equiv (\hbar^2/2m)\psi^*_{xx} - i\hbar\psi^*_t - U\psi^* \end{aligned}\right\} \tag{70}$$

and find it natural to ask: Does there exist a Lagrangian density $\mathfrak{L}$ such that

$$\left.\begin{aligned} G_1 &= \left\{ \frac{\partial}{\partial t}\frac{\partial}{\partial \psi_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \psi_x} - \frac{\partial}{\partial \psi} \right\}\mathfrak{L} \\ G_2 &= \left\{ \frac{\partial}{\partial t}\frac{\partial}{\partial \psi^*_t} + \frac{\partial}{\partial x}\frac{\partial}{\partial \psi^*_x} - \frac{\partial}{\partial \psi^*} \right\}\mathfrak{L} \end{aligned}\right\} \ ? \tag{71}$$

It was to answer just such questions that the Helmholtz conditions (44) were devised. In the present context

$$N \equiv \text{number of field components} = 2$$
$$m \equiv \text{number of spacetime dimensions} = 2$$

so these are—rather soberingly— $3 + 6 + 1 = 10$ in number. I write them out:

$$\frac{\eth G_1}{\eth \psi^*_{tt}} - \frac{\eth G_2}{\eth \psi_{tt}} = 0 \tag{72.1}$$
$$\frac{\eth G_1}{\eth \psi^*_{tx}} - \frac{\eth G_2}{\eth \psi_{tx}} = 0 \tag{72.2}$$
$$\frac{\eth G_1}{\eth \psi^*_{xx}} - \frac{\eth G_2}{\eth \psi_{xx}} = 0 \tag{72.3}$$

$$\frac{\partial G_1}{\partial \psi_t} + \frac{\partial G_1}{\partial \psi_t} = \frac{\partial}{\partial t}\left[\frac{\eth G_1}{\eth \psi_{tt}} + \frac{\eth G_1}{\eth \psi_{tt}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_1}{\eth \psi_{xt}} + \frac{\eth G_1}{\eth \psi_{xt}}\right] \tag{73.1}$$
$$\frac{\partial G_1}{\partial \psi^*_t} + \frac{\partial G_2}{\partial \psi_t} = \frac{\partial}{\partial t}\left[\frac{\eth G_1}{\eth \psi^*_{tt}} + \frac{\eth G_2}{\eth \psi_{tt}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_1}{\eth \psi^*_{xt}} + \frac{\eth G_2}{\eth \psi_{xt}}\right] \tag{73.2}$$
$$\frac{\partial G_2}{\partial \psi^*_t} + \frac{\partial G_2}{\partial \psi^*_t} = \frac{\partial}{\partial t}\left[\frac{\eth G_2}{\eth \psi^*_{tt}} + \frac{\eth G_2}{\eth \psi^*_{tt}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_2}{\eth \psi^*_{xt}} + \frac{\eth G_2}{\eth \psi^*_{xt}}\right] \tag{73.3}$$
$$\frac{\partial G_1}{\partial \psi_x} + \frac{\partial G_1}{\partial \psi_x} = \frac{\partial}{\partial t}\left[\frac{\eth G_1}{\eth \psi_{tx}} + \frac{\eth G_1}{\eth \psi_{tx}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_1}{\eth \psi_{xx}} + \frac{\eth G_1}{\eth \psi_{xx}}\right] \tag{73.4}$$
$$\frac{\partial G_1}{\partial \psi^*_x} + \frac{\partial G_2}{\partial \psi_x} = \frac{\partial}{\partial t}\left[\frac{\eth G_1}{\eth \psi^*_{tx}} + \frac{\eth G_2}{\eth \psi_{tx}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_1}{\eth \psi^*_{xx}} + \frac{\eth G_2}{\eth \psi_{xx}}\right] \tag{73.5}$$
$$\frac{\partial G_2}{\partial \psi^*_x} + \frac{\partial G_2}{\partial \psi^*_x} = \frac{\partial}{\partial t}\left[\frac{\eth G_2}{\eth \psi^*_{tx}} + \frac{\eth G_2}{\eth \psi^*_{tx}}\right] + \frac{\partial}{\partial x}\left[\frac{\eth G_2}{\eth \psi^*_{xx}} + \frac{\eth G_2}{\eth \psi^*_{xx}}\right] \tag{73.6}$$

$$\frac{\partial G_1}{\partial \psi^*} - \frac{\partial G_2}{\partial \psi} = \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial G_1}{\partial \psi^*_t} - \frac{\partial G_2}{\partial \psi_t}\right] + \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial G_1}{\partial \psi^*_x} - \frac{\partial G_2}{\partial \psi_x}\right] \tag{74.1}$$

Taking the G_1 and G_2 as our input, we find by quick calculation that eight of these ten conditions are satisfied—indeed, are trivially satisfied—but that (73.1) and (73.3) are *not* satisfied, for they entail the absurdity $i\hbar = 0$. We are, however, well aware (see again the discussion of this point on p. 34) that the Helmholtz test is (like computer software tends to be) utterly unforgiving of little misalignments of the input data. So enticed are we by the attractiveness of our paradoxical objective, and so encouraged by the knowledge that the closely related forward/backward diffusion equations *can* be made to pass the test, that we experiment a bit... and are led soon enough to *interchange the definitions* (70) of the functions G; we define

$$\left.\begin{aligned} G_2 &\equiv (\hbar^2/2m)\psi_{xx} + i\hbar\psi_t - U\psi \\ G_1 &\equiv (\hbar^2/2m)\psi^*_{xx} - i\hbar\psi^*_t - U\psi^* \end{aligned}\right\} \tag{75}$$

and try again. A few seconds of work is sufficient to establish that the functions G_1 and G_2—thus defined—*do* pass all parts of the Helmholtz test. Thus encouraged to think that it *may* be possible to write (71), we do a little inspired tinkering[37] and are led at length to an $\mathcal{L}$ that works:

$$\mathcal{L} = \tfrac{1}{2}i\hbar(\psi^*_t\psi - \psi^*\psi_t) + \tfrac{\hbar^2}{2m}\psi^*_x\psi_x + \psi^*U\psi \tag{76}$$

The $\mathcal{L}_{\text{Schrödinger}}$ of (76) possesses several striking and important properties to which I would like now to draw attention. It is, in the first place, *manifestly real*

$$\mathcal{L}^* = \mathcal{L}$$

and so also, therefore, is the associated action functional $S_{\mathcal{R}}[\psi] = \int \mathcal{L}\, dxdt$. This is gratifying, for in the contrary case we would have in hand a "multi-component Langrangian" $\mathcal{L} = \mathcal{L}_1 + i\mathcal{L}_2$, and so would have acquired an obligation to venture in to some highly non-standard formal territory.[38] We might also expect (for reasons basic to the theory of functions of several complex variables) to encounter difficulties in giving precise meaning to Hamilton's Principle $\delta S = 0$. Note also the "bilinear" structure of $\mathcal{L}_{\text{Schrödinger}}$; it is (manifestly) linear in the variables $\{\psi, \psi_x, \psi_t\}$ and linear also in the complex conjugates of those variables. Bilinearity accounts for the uncoupled linearity of the Schrödinger equation (69). We observe in passing that if we write $\psi = \varphi^1 + i\varphi^2$ then

[37] In the real world of work-a-day physics one is well-advised to do one's "inspired tinkering" *at the outset*; i.e., to skip the Helmholtzian folderol, which I have written out only for didactic effect. Helmholtzian methods of greatest practical utility when used to establish the *impossibility* of casting the field equations of momentary interest in Lagrangian form; in such applications they alert one to the need to "do something heroic."

[38] See, in this connection, T. Morgan & D. Joseph, "Tensor Lagrangians and generalized conservation laws for free fields," Nuovo Cimento, **39**, 494 (1965) and the literature cited.

$$\mathfrak{L} = \hbar(\varphi^1\varphi^2_t - \varphi^1_t\varphi^2) + \tfrac{\hbar^2}{2m}(\varphi^1_x\varphi^1_x + \varphi^2_x\varphi^2_x) + U(\varphi^1\varphi^1 + \varphi^2\varphi^2) \tag{77}$$

while $\psi = R\exp\left\{\frac{i}{\hbar}S\right\}$ gives

$$\mathfrak{L} = R^2\left[S_t + \tfrac{1}{2m}S_x^2 + U\right] + \tfrac{\hbar^2}{2m}R_x^2 \tag{78}$$

and that the field-variable adjustments which achieve (77) and (78), while they preserve the reality of (76), do violence to its bilinearity. Commentary concerning what is, from some points of view, the most interesting feature of $\mathfrak{L}_{\text{Schrödinger}}$ (namely, that it has actually the status of a *meta*-Lagrangian) will, however, be reserved until the point at issue has been more carefully framed.

Schrödinger's Lagrangian (76) gives rise (by Noetherian analysis) to a stress-energy tensor the components of which can, by (34), be described

$$\begin{aligned}
S^t{}_t &= \frac{\partial\mathfrak{L}}{\partial\psi_t}\,\psi_t + \frac{\partial\mathfrak{L}}{\partial\psi^*_t}\,\psi^*_t - \mathfrak{L} \\
&= -\tfrac{\hbar^2}{2m}\psi^*_x\psi_x - \psi^*U\psi && (79.1)\\
S^x{}_t &= \frac{\partial\mathfrak{L}}{\partial\psi_x}\,\psi_t + \frac{\partial\mathfrak{L}}{\partial\psi^*_x}\,\psi^*_t \\
&= \tfrac{\hbar^2}{2m}(\psi^*_x\psi_t + \psi^*_t\psi_x) && (79.2)\\
S^t{}_x &= \frac{\partial\mathfrak{L}}{\partial\psi_t}\,\psi_x + \frac{\partial\mathfrak{L}}{\partial\psi^*_t}\psi^*_x \\
&= \tfrac{1}{2}i\hbar(\psi^*_x\psi - \psi^*\psi_x) && (79.3)\\
S^x{}_x &= \frac{\partial\mathfrak{L}}{\partial\psi_x}\psi_x + \frac{\partial\mathfrak{L}}{\partial\psi^*_x}\psi^*_x - \mathfrak{L} \\
&= -\tfrac{1}{2}i\hbar(\psi^*_t\psi - \psi^*\psi_t) + \tfrac{\hbar^2}{2m}\psi^*_x\psi_x - \psi^*U\psi && (79.4)
\end{aligned}$$

By straightforward calculation (which draws critically upon the field equations (69); i.e., upon the Schrödinger equation and its conjugate) we find

$$\partial_t S^t{}_t + \partial_x S^x{}_t = -\psi^*\frac{\partial U}{\partial t}\psi \tag{80.1}$$

$$\partial_t S^t{}_x + \partial_x S^x{}_x = -\psi^*\frac{\partial U}{\partial x}\psi \tag{80.2}$$

These equations speak quite intelligibly to us (in language quite consistent with our Noetherian experience): the former says that

> *Energy will be conserved unless the t-dependence of U serves to break the time-translational invariance of the system*,

while the latter says that

> *Momentum will be conserved unless the x-dependence of U serves to break the space-translational invariance of the system.*

Of course, we do not expect momentum to be conserved in the presence of forces $F = -\partial U/\partial x$!

Equations (80) are local statements, very much in the Lagrangian tradition of classical field theory (though we are at present doing the *quantum* mechanics of a *particle*!), and put us in position to state that the total energy of our field system—an "interesting quantity," whether conserved or not—can be described[39]

$$\begin{aligned} E &= \int (-S^t{}_t)\, dx \\ &= \int \left\{ \tfrac{\hbar^2}{2m} \psi^*_x \psi_x + \psi^* U \psi \right\} dx \\ &= \int \left\{ -\tfrac{\hbar^2}{2m} \psi^* \psi_{xx} + \psi^* U \psi \right\} dx \end{aligned}$$

where to achieve the final result I have integrated by parts and discarded the boundary term. Evidently we can, if we wish, write

$$E = \int \psi^* \left\{ \tfrac{1}{2m} \left(\tfrac{\hbar}{i} \tfrac{\partial}{\partial x} \right)^2 + U \right\} \psi \, dx \tag{81}$$

Similarly, the total momentum can be described

$$\begin{aligned} p &= \int S^t{}_x \, dx \\ &= \int \tfrac{1}{2} i\hbar (\psi^*_x \psi - \psi^* \psi_x)\, dx \\ &= -\int i\hbar (\psi^* \psi_x)\, dx \\ &= \int \psi^* \left\{ \left(\tfrac{\hbar}{i} \tfrac{\partial}{\partial x} \right) \right\} \psi \, dx \end{aligned} \tag{82}$$

Readers will not fail to notice that the expressions which stand on the right sides of equations (81) and (82) are precisely the expressions which in quantum mechanics are taken to describe $\langle E \rangle_\psi$ and $\langle p \rangle_\psi$—the *expected values* of energy and momentum, given that the particle is in state ψ. As field theorists we are struck by the fact (which is for us as quantum mechanics already old news!) that (82) can be notated

$$p = \int \psi^* \mathsf{p} \, \psi \, dx \quad \text{where } \mathsf{p} \text{ is the } \textit{differential operator } \mathsf{p} \equiv \tfrac{\hbar}{i} \tfrac{\partial}{\partial x} \tag{83}$$

and that in this notation (81) becomes

$$E = \int \psi^* \mathsf{H} \, \psi \, dx \quad \text{with} \quad \mathsf{H} \equiv \tfrac{1}{2m} \mathsf{p}^2 + U(\mathsf{x}) \tag{84}$$

while (66)—the Schrödinger equation itself—becomes

$$\mathsf{H}\psi = i\hbar \psi_t$$

[39] I exercise here my freedom to introduce a minus sign in order to achieve consistency with notationial and interpretive orthodoxies later on.

The point I have been trying in recent remarks to illustrate is this: if one looks upon the Schrödinger equation with the eyes of a classical field theorist, then one is led *naturally/spontaneously* to the "quantization procedure"

$$\begin{aligned} p &\longrightarrow \mathsf{p} = \tfrac{\hbar}{i}\tfrac{\partial}{\partial x} \\ x &\longrightarrow \mathsf{x} = x\cdot \end{aligned}$$

and to the formulae (such, for example, as (83) and (84)) most characteristic of the quantum theory of particulate motion.

From the (previously noted) bilinear reality of (76) it follows that $\mathfrak{L}$ —whence also $S = \int \mathfrak{L} dx$ —is invariant under "phase transformations" of the form

$$\left.\begin{aligned} \psi &\longrightarrow e^{+i\omega}\psi \\ \psi^* &\longrightarrow e^{-i\omega}\psi^* \end{aligned}\right\} \tag{85.1}$$

and therefore (we note in passing) that (77) is invariant under

$$\left.\begin{aligned} \varphi^1 &\longrightarrow \varphi^1\cos\omega - \varphi^2\sin\omega \\ \varphi^2 &\longrightarrow \varphi^1\sin\omega + \varphi^2\cos\omega \end{aligned}\right\} \tag{85.2}$$

and (78) invariant under

$$\left.\begin{aligned} R &\longrightarrow R \\ S &\longrightarrow S + \hbar\omega \end{aligned}\right\} \tag{85.3}$$

To describe an *infinitesimal* phase transformation we write

$$\left.\begin{aligned} t &\longrightarrow t + \delta_\omega t \quad \text{with} \quad \delta_\omega t = 0\cdot\delta\omega \\ x &\longrightarrow x + \delta_\omega x \quad \text{with} \quad \delta_\omega x = 0\cdot\delta\omega \\ \psi &\longrightarrow \psi + \delta_\omega\psi \quad \text{with} \quad \delta_\omega\psi = +i\psi\cdot\delta\omega \\ \psi^* &\longrightarrow \psi^* + \delta_\omega\psi^* \quad \text{with} \quad \delta_\omega\psi^* = -i\psi^*\cdot\delta\omega \end{aligned}\right\} \tag{86}$$

We have encountered here our first example of a purely "*internal* symmetry" of a multi-component field system; the fields are "folding among themselves," but nothing else is going on. The field variations derive (in the language of p. 24) entirely from "variation of functional form," and not at all from "variation of argument." To the extent that we find (84) an "interesting map" we should, according to Noether, have interest in the current whose components can, by (29), be described

$$\begin{aligned} P^t &= \frac{\partial\mathfrak{L}}{\partial\psi_t}(i\psi) + \frac{\partial\mathfrak{L}}{\partial\psi_t^*}(-i\psi^*) \\ &= \hbar\cdot\psi^*\psi \end{aligned} \tag{87.1}$$

$$\begin{aligned} P^x &= \frac{\partial\mathfrak{L}}{\partial\psi_x}(i\psi) + \frac{\partial\mathfrak{L}}{\partial\psi_x^*}(-i\psi^*) \\ &= \hbar\cdot i\tfrac{\hbar}{2m}(\psi_x^*\psi - \psi^*\psi_x) \end{aligned} \tag{87.2}$$

and we should not be surprised to discover that (in consequence of the field equations)

$$\partial_t P^t + \partial_x P^x = 0$$

The global formulation of this local conservation law reads

$$\int \psi^* \psi \, dx = \text{constant of the field motion} \tag{89}$$

This result acquires interest from several distinct considerations. It shows, in the first place, that no risk of inconsistency was incurred when, at (65), we imposed the normalization condition $\int \psi^* \psi \, dx = 1$. While (89) speaks to our physical intuitions (as historically it spoke in 1925 to Schrödinger himself) of the conservation of some kind of "charge," it does *not* carry within it any hint of anything having to do with a "statistical interpretation of the wave function." It permits—but certainly does not force—us to write (with Born, 1926)

$$\begin{aligned} \text{"probability density"} &= \psi^* \psi \\ &= \varphi^1 \varphi^1 + \varphi^2 \varphi^2 \\ &= R^2 \end{aligned} \tag{90.1}$$

or (collaterally)

$$\begin{aligned} \text{"probability current"} &= i \tfrac{\hbar}{2m} (\psi_x^* \psi - \psi^* \psi_x) \\ &= m \cdot \text{momentum density} \\ &= \tfrac{\hbar}{m} (\varphi_x^2 \varphi^1 - \varphi^2 \varphi_x^1) \\ &= \tfrac{1}{m} S_x R^2 \end{aligned} \tag{90.2}$$

We are in position now to notice, by the way, that the conservation theorem

$$\tfrac{\partial}{\partial t}(\text{probability density}) + \tfrac{\partial}{\partial x}(\text{probability current}) = 0 \tag{91}$$

lends direct physical significance to the 2nd of the "polar Schrödinger equations" (68). It is, finally, of deep interest (and a harbinger of beautiful things to come) that (91) is attributed within the field-theoretic formulation of quantum mechanics to an *internal symmetry* of the theory.

The quantum theory of a particle is (at least as standardly formulated) so profoundly "Hamiltonian" in spirit that we, as field theorists, find it natural to ask: Do the field equations admit of Hamiltonian formulation within $\mathcal{H}^t$? within $\mathcal{H}^x$? Working first within $\mathcal{H}^t$, we look to (76) and obtain

$$\begin{aligned} \text{momentum conjugate to } \psi &= -\tfrac{1}{2} i\hbar \psi^* \\ \text{momentum conjugate to } \psi^* &= +\tfrac{1}{2} i\hbar \psi \end{aligned}$$

Evidently

$$\mathcal{L} \xrightarrow[\text{Legendre transformation}]{} \mathcal{H}$$

is *not possible*, for reasons of precisely the sort discussed already on p. 57. Nor do the variant Lagrangians (77) and (78) afford, in this regard, any advantages. But... suppose we write

$$\pi = \text{momentum conjugate to } \psi = -\tfrac{1}{2}i\hbar\psi^* \tag{92}$$

—which is to make of ψ^* a field "conjugate" to ψ in quite a novel sense— and construe (76) to have the character of a *meta*-Lagrangian, writing (after multiplication by an inconsequential $\frac{1}{2}$)

$$\mathfrak{L} = \tfrac{1}{2}(\pi\psi_t - \pi_t\psi) + \tfrac{i}{\hbar}(\tfrac{\hbar^2}{2m}\pi_x\psi_x + \pi U\psi) \tag{93}$$

Certainly this meta-Lagrangian works, in the easily confirmed sense that

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\mathfrak{L}}{\partial\pi_t} + \frac{\partial}{\partial x}\frac{\partial\mathfrak{L}}{\partial\pi_x} - \frac{\partial\mathfrak{L}}{\partial\pi} &= 0\\ \frac{\partial}{\partial t}\frac{\partial\mathfrak{L}}{\partial\psi_t} + \frac{\partial}{\partial x}\frac{\partial\mathfrak{L}}{\partial\psi_x} - \frac{\partial\mathfrak{L}}{\partial\psi} &= 0\end{aligned}$$

do reproduce the Schrödinger equations (69), which we would now notate

$$\left.\begin{aligned}-\tfrac{\hbar^2}{2m}\psi_{xx} + U\psi &= +i\hbar\psi_t\\ -\tfrac{\hbar^2}{2m}\pi_{xx} + U\pi &= -i\hbar\pi_t\end{aligned}\right\} \tag{94}$$

Comparison of (93) with (63)—which it so closely resembles—leads us to define

$$\mathcal{H}(\pi,\psi) \equiv -\tfrac{i}{\hbar}(\tfrac{\hbar^2}{2m}\pi_x\psi_x + \pi U\psi) \tag{95}$$

and to observe that the implied canonical field equations

$$\left.\begin{aligned}\psi_t &= +\left\{\frac{\partial}{\partial\pi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\pi_x}\right\}\mathcal{H} = -\tfrac{i}{\hbar}(U\psi - \tfrac{\hbar^2}{2m}\psi_{xx})\\ \pi_t &= -\left\{\frac{\partial}{\partial\psi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\psi_x}\right\}\mathcal{H} = +\tfrac{i}{\hbar}(U\pi - \tfrac{\hbar^2}{2m}\pi_{xx})\end{aligned}\right\} \tag{96}$$

again reproduce the Schrödinger equations (94). So quantum mechanics *does*, within $\mathcal{H}^t$, admit of formulation as a Hamiltonian field theory. Note, however, that the Hamiltonian (95) is "free-standing" in the sense that it does not admit of

$$\mathcal{L} \underset{\text{Legendre transformation}}{\longleftarrow\!\longrightarrow} \mathcal{H}$$

What we took at (76) to be a description of $\mathcal{L}_{\text{Schrödinger}}$ was, we now see, actually a deceptively notated description of the meta-Lagrangian $\mathfrak{L}_{\text{Schrödinger}}$.

Within Hamiltonian formalism we attribute the occurance of conservation laws to the special properties acquired by certain Poisson brackets. For example, we might notice that

$$\text{"probability density"} \equiv \psi^*\psi = 2\tfrac{i}{\hbar}\cdot\pi\psi$$

and compute

$$
\begin{aligned}
[\,\pi\psi, \mathcal{H}\,] &= \left\{ \frac{\partial \mathcal{R}}{\partial \psi} - \frac{\partial}{\partial x}\frac{\partial \mathcal{R}}{\partial \psi_x} \right\} \left\{ \frac{\partial \mathcal{H}}{\partial \pi} - \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial \pi_x} \right\} \\
&\qquad - \left\{ \frac{\partial \mathcal{H}}{\partial \psi} - \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial \psi_x} \right\} \left\{ \frac{\partial \mathcal{R}}{\partial \pi} - \frac{\partial}{\partial x}\frac{\partial \mathcal{R}}{\partial \pi_x} \right\} \quad \text{with } \mathcal{R} \equiv \pi\psi \\
&= -\tfrac{i}{\hbar} \Big\{ \pi \Big(U\psi - \tfrac{\hbar^2}{2m}\psi_{xx} \Big) - \Big(\pi U - \tfrac{\hbar^2}{2m}\pi_{xx} \Big)\psi \Big\} \\
&= -\tfrac{i}{\hbar}\tfrac{\hbar^2}{2m}(\pi_{xx}\psi - \pi\psi_{xx}) \\
&= -\tfrac{\partial}{\partial x} \Big\{ i\tfrac{\hbar}{2m}(\pi_x\psi - \pi\psi_x) \Big\} \qquad (97)
\end{aligned}
$$

Therefore (discarding a boundary term) $\frac{\partial}{\partial t}\int \pi\psi\, dx = 0$, which is the upshot of (89). We note that the expression interior to the final braces is proportional to "probability current," as described in (90.1). And that if we bring the equations of motion (94) to the braces in the second equation we obtain $[\,\pi\psi, \mathcal{H}\,] = (\pi\psi)_t$; we have in fact obtained by Hamiltonian means an equation which differs from (91) only by an overall factor.

The discovery that the Lagrangian $\mathcal{L}_{\text{Schrödinger}}$ is (at least within $\mathcal{H}^t$) more properly construed to be a meta-Lagrangian carries within it the seed of a very important lesson. For it means that when we successfully (and quite informatively) brought Noetherian methods to bear upon $\mathcal{L}_{\text{Schrödinger}}$ we were actually bringing those methods to bear upon a *meta*-Lagrangian, and that's a much more general object; the arguments of a meta-Lagrangian range not on configuration space but on phase space (which accounts for the fact that such objects are most commonly encountered in connection with the theory of canonical transformations). By (unwittingly) transposing Noether's line of argument from Lagrangian physics to meta-Lagrangian physics we have in effect created a vast *generalization of Noether's method*, as standardly conceived. And since (if I may lapse for a moment into the simpler language of particle mechanics)

$$\mathbf{L}(\dot{p}, \dot{q}, p, q) = \tfrac{1}{2}(p\dot{q} - \dot{p}q) - H(p, q)$$

exists independently of whether or not $H(p, q)$ happens to be "free-standing" (even, that is to say, when an associated $L(\dot{q}, q)$ cannot be constructed), we have in fact resolved the problem which (at the bottom of p. 57) we have described as an "invitation to invention."

We have been witnesses to a conversation between classical field theory and quantum mechanics that seemed a moment ago to be at the point of wrapping up, to the satisfaction of all participants... but which now takes a new turn. For suppose we elect to work (not, as above, within $\mathcal{H}^t$ but) within $\mathcal{H}^x$. We look back again to (76) and obtain

$$\pi \equiv \text{momentum conjugate to } \psi = \tfrac{\hbar^2}{2m}\psi_x^* \qquad (98.1)$$

$$\pi^* \equiv \text{momentum conjugate to } \psi^* = \tfrac{\hbar^2}{2m}\psi_x \qquad (98.2)$$

and observe that it now *is* possible to proceed

$$\mathcal{L} \xrightarrow[\text{Legendre transformation}]{\qquad\qquad\qquad\qquad} \mathcal{H}$$

We write

$$\begin{aligned}\mathcal{H} &= \pi\psi_x + \pi^*\psi_x^* - \mathcal{L} \\ &= \pi\psi_x + \pi^*\psi_x^* - \tfrac{1}{2}i\hbar(\psi_t^*\psi - \psi^*\psi_t) - \tfrac{\hbar^2}{2m}\psi_x^*\psi_x - \psi^*U\psi\end{aligned}$$

and by substitutions $\psi_x \longrightarrow \frac{2m}{\hbar^2}\pi^*$ and $\psi_x^* \longrightarrow \frac{2m}{\hbar^2}\pi$ obtain

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(\pi, \pi^*, \psi, \psi^*) \\ &= \tfrac{2m}{\hbar^2}\pi^*\pi - \psi^*U\psi - \tfrac{1}{2}i\hbar(\psi_t^*\psi - \psi^*\psi_t)\end{aligned} \tag{99}$$

The canonical field equations are now four in number:

$$\psi_x = +\left\{\frac{\partial}{\partial\pi} - \frac{\partial}{\partial t}\frac{\partial}{\partial\pi_t}\right\}\mathcal{H} = \tfrac{2m}{\hbar^2}\pi^* \tag{100.1}$$

$$\pi_x = -\left\{\frac{\partial}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial}{\partial\psi_t}\right\}\mathcal{H} = U\psi^* + i\hbar\psi_t^* \tag{100.2}$$

$$\psi_x^* = +\left\{\frac{\partial}{\partial\pi^*} - \frac{\partial}{\partial t}\frac{\partial}{\partial\pi_t^*}\right\}\mathcal{H} = \tfrac{2m}{\hbar^2}\pi \tag{100.3}$$

$$\pi_x^* = -\left\{\frac{\partial}{\partial\psi^*} - \frac{\partial}{\partial t}\frac{\partial}{\partial\psi_t^*}\right\}\mathcal{H} = U\psi - i\hbar\psi_t \tag{100.4}$$

The first and third of these equations give back (98). The 1st order canonical equations collectively reproduce precisely the 2nd order Schrödinger equations (94). It should be noticed that when we shift from one variant of Hamiltonian formalism to another we find ourselves writing equations that are superficially identical, but which carry quite distinct meanings, with the result that the associated calculations which can sometimes feel radially different. For example, to account for "probability conservation" (89) within $\mathcal{H}^x$ we compute

$$\begin{aligned}[\psi^*\psi, \mathcal{H}] = &\left\{\frac{\partial\mathcal{R}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{R}}{\partial\psi_t}\right\}\left\{\frac{\partial\mathcal{H}}{\partial\pi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{H}}{\partial\pi_t}\right\} \\ &+ \left\{\frac{\partial\mathcal{R}}{\partial\psi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{R}}{\partial\psi_t^*}\right\}\left\{\frac{\partial\mathcal{H}}{\partial\pi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{H}}{\partial\pi_t^*}\right\} \\ &- \left\{\frac{\partial\mathcal{H}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{H}}{\partial\psi_t}\right\}\left\{\frac{\partial\mathcal{R}}{\partial\pi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{R}}{\partial\pi_t}\right\} \\ &- \left\{\frac{\partial\mathcal{H}}{\partial\psi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{H}}{\partial\psi_t^*}\right\}\left\{\frac{\partial\mathcal{R}}{\partial\pi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal{R}}{\partial\pi_t^*}\right\} \quad \text{with } \mathcal{R} \equiv \psi^*\psi \\ = \psi^*\tfrac{\hbar^2}{2m}\pi^* + \psi\tfrac{\hbar^2}{2m}\pi\end{aligned}$$

which by (100.1) and (100.3) can be written

$$= \tfrac{\partial}{\partial x}(\psi^*\psi)$$

This is interesting information, but does not have the form $\frac{\partial}{\partial t}$(something) which

is essential to the establishment of global conservation theorems within the $\mathcal{H}^x$ formalism. But of course! When the Lagrange formalism supplies $\frac{\partial}{\partial t}\rho+\boldsymbol{\nabla}\cdot\boldsymbol{J}=0$ we look within $\mathcal{H}^t$ to $[\rho,\mathcal{H}]$, but within $\mathcal{H}^x$ should look to the x-component of $[\boldsymbol{J},\mathcal{H}]$. Now

$$x\text{-component of ``probability current''} = i\tfrac{\hbar^2}{2m}(\psi^*_x\psi - \psi_x\psi^*)$$

according to (90.2), so drawing upon (98) we have

$$= \tfrac{i}{\hbar}(\pi\psi - \pi^*\psi^*)$$

and by quick computation obtain

$$[\,x\text{-component of ``probability current''},\mathcal{H}\,] = -\tfrac{\partial}{\partial t}(\psi^*\psi) \tag{101}$$

which has precisely the anticipated form. The computational experience which led us to (101) is vividly distinct from that which led us to (97). Similarly distinct—see again FIGURE 8—are the meanings of the conservation theorems thus achieved. For while (97) refers to the invariance of data written onto t-sections in spacetime (and is therefore a "conservation theorem" in the familiar sense), (101) refers to the invariance of (other) data written onto x-sections.

A few comments before I bring this discussion to a close: I begin by drawing attention to the familiar fact that quantum theory is a wonderfully *abstract* physical theory; only by abandoning some of that abstraction do we find ourselves working in the "Schrödinger picture"

$$H|\psi\rangle = i\hbar\tfrac{\partial}{\partial t}|\psi\rangle$$

and only by further arbitrary acts—for example, by electing to work in the x-representation

$$\psi(x) = \langle x|\psi\rangle$$

or perhaps in the associated "Wigner representation"[40]

$$P_\psi(p,x) = \tfrac{2}{h}\int \psi^*(x+\xi)e^{2\frac{i}{\hbar}p\xi}\psi(x-\xi)\,d\xi$$

—does quantum mechanics acquire the status of a field theory. It seems to me unreasonable—unacceptable—that classical field theory should be able to "see," and to comment usefully upon, quantum mechanics only when the latter subject is suitably costumed; if classical field theory has things to say about quantum mechanics, it should be able to say them about "naked" quantum mechanics. The patient-analyst roles have at this point been reversed; quantum mechanics is telling us that to understand her we should first attend to the improvement of classical field theory, at least as it relates to *linear* systems, and (if I read her correctly) that we should attend in particular to the canonical transform

[40] See QUANTUM MECHANICS (1967), Chapter III, pp. 99 *et seq*.

aspects of classical field theory. For it is by transformation theory that we learn to look beyond the representations of things to the structural essentials of things-in-themselves.[41]

Our quantum experience has—relatedly, I think—also alerted us to the richness of the interconnections amongst the Lagrangian, Hamiltonian and meta-Lagrangian modes of conceptualizing the dynamics of specific classical field systems. Recall the pattern of that experience: we started with the $\mathcal{L}$ of (76) and found that

$$\mathcal{L} \longrightarrow \begin{cases} \mathcal{H}^t & \text{is not possible, while} \\ \mathcal{H}^x & \text{is possible} \end{cases}$$

We noticed that a Hamiltonian quantum mechanics of type $\mathcal{H}^t$ did, however, become possible if $\mathcal{L}$ were assigned the meaning of a meta-Lagrangian $\boldsymbol{\mathcal{L}}$; we then obtained this "folliation" of the initial Lagrangian theory:

$$\begin{array}{ccccc} & & \mathcal{H} \text{ within } \mathcal{H}^t & \longleftarrow & \boldsymbol{\mathcal{L}} \\ & \ulcorner & \mathcal{L} & \lrcorner & \\ \mathcal{L} & \longrightarrow & \mathcal{H} \text{ within } \mathcal{H}^x & \longrightarrow & \tilde{\boldsymbol{\mathcal{L}}} \end{array}$$

where

$$\tilde{\boldsymbol{\mathcal{L}}} = \pi\psi_x + \pi^*\psi_x^* - \left\{ \tfrac{2m}{\hbar^2}\pi^*\pi - \psi^* U \psi - \tfrac{1}{2} i\hbar(\psi_t^*\psi - \psi^*\psi_t) \right\}$$

While the phase space associated with the $\mathcal{H}^t$ formalism is 2-dimensional, that associated with the $\mathcal{H}^x$ formalism is 4-dimensional, and might before be expected to support a richer population of symmetry relations; those we might expect to expose (pursuant to a point developed on p. 68) by bringing Noetherian methods to bear on $\tilde{\boldsymbol{\mathcal{L}}}$.

Classical field theory of the Hamilton-Jacobi equation. Schrödinger, as is well known, was led to the equation which bears his name by an avowedly analogical procedure: building upon speculation (1923) of DeBroglie ("the mechanics of particles is in some respects wave-like"), who was himself inspired by Einstein's demonstration (1905) that "the mechanics of electromagnetic radiation is in some respects particle-like," Schrödinger sought (in his phrase) to "complete the optico-mechanical analogy"

$$\begin{array}{ccc} \text{geometrical optics} & \longleftarrow & \text{physical optics} \\ \downarrow & & \downarrow \\ \text{classical mechanics} & \longrightarrow & \boxed{\text{wave mechanics}} \end{array}$$

[41] It was, in fact, precisely thus that quantum mechanics itself *acquired* its abstract identity; see M. Jammer, The Conceptual Development of Quantum Mechanics (1966)

That geometrical optics (the optics of "rays"—a subject implicit in the variational principle of Fermat) and classical mechanics (which addresses the design of the "trajectories" temporally traced by mass points, and is implicit in the variational principle of Lagrange) are homologous structures had been established already by Hamilton, nearly a century before (1830–1834), and rediscovered by Bruns.[42] The idea that geometrical optics is an artifact of an underlying theory of "optical waves"—and becomes available only in a certain approximation[43]—is, of course, so ancient that Newton was at pains to "refute" it, and by the last third of the 19th Century had been brought to a high state of development; Fresnel, Helmholtz, Kirchhoff and others—working in pursuit of an idea original to Huygens[44]—had erected a "physical optics" in which a "wave equation" was the fundamental analytical device, and diffraction, superposition (linearity), interference and polarization were among the most characteristic phenomenological manifestations. In the early 1920's—immediately prior to Schrödinger's involvement—it had become evident that those same phenomena are manifest in the mechanics of small particles. When Schrödinger sought to account for this development by "completion of the optico-mechanical analogy" his more precise objective was to devise a "wave mechanics" which stands to Hamiltonian mechanics (Hamilton-Jacobi theory) as physical optics stands to Hamiltonian optics; it was to the most rarified (and—then as now—least widely

[42] Hamilton himself proceeded from the observation that his formulation of geometrical optics was adaptable to mechanics, where it gave rise to the "canonical equations of motion," the theory of canonical transformations and what we now call "Hamilton-Jacobi theory" (Jacobi entered the picture in 1837). His mechanical work became widely known, but its optical precursor fell rapidly into obscurity—especially in Europe. Almost seventy years were to pass before Heinrich Bruns—a student of Sophus Lie, from whom he had acquired a sensitivity to the *geometry* of differential equations—reported his "discovery" ("Das Eikonal," Sächs. Ber. D. Wiss. **21**,1895) that Hamilton's mechanics could be adapted to the needs of geometrical optics. Contemporary optical theorists generally claim desent from Hamilton, rather than Bruns, but have appropriated Bruns' "eikonal" terminology when referring to what Hamilton called the "characteristic function." Bruns' work, though redundant, did serve to revive interest in Hamiltonian optics, and served to underscore the fact that—on the classical side of the ledger—the "optico-mechanical analogy" refers to an association of the form

$$\text{Hamiltonian optics} \quad \longleftrightarrow \quad \text{Hamiltonian mechanics}$$

[43] What approximation? The essential clue is provided by the fact that while the concept of "ray" is recommended to our intuition by the evidence of our eyes, our acoustic experience does not make it natural to speak of "sound rays."

[44] *Traité de la lumière* (1690). For a suberb account of the developments to which I allude, see B. B. Baker & E. T. Copson, *The Mathematical Theory of Huygens' Principle* (1950).

understood) formulation of classical mechanics that Schrödinger looked for his point of departure.[45] More specifically...

The classical object to which Schrödinger looked for quantum guidance was the Hamilton-Jacobi equation, which in the simplest instance (mass point m moving one-dimensionally in the presence of an impressed potential) can be written

$$\tfrac{1}{2m}S_x^2 + U + S_t = 0 \quad : \quad S \equiv S(x,t) \tag{102.1}$$

and in progressively more general classes of cases becomes

$$\begin{array}{rcll}
\tfrac{1}{2m}\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}S + U + S_t = 0 & : & S \equiv S(\boldsymbol{x},t) & : \quad \text{relax one-dimensionality} \\
& \downarrow & & \\
H(\boldsymbol{\nabla}S, \boldsymbol{x}) + S_t = 0 & : & \text{relax restriction on structure of } H(\boldsymbol{p},\boldsymbol{x}) & \\
& \downarrow & & \\
H\big(\tfrac{\partial S}{\partial q}, q\big) + \tfrac{\partial S}{\partial t} = 0 & : & S \equiv S(q,t) \equiv S(\underbrace{q^1, q^2, \ldots, q^n}_{\text{generalized coordinates}}, t) &
\end{array} \tag{102.2}$$

It is a remarkable fact that that the classical mechanics of a particle (or system of particles), when formulated at such a high level[46] of abstraction, is described no longer by a system of coupled ordinary differential equations, but by a single *partial* differential equation; it has, in short become a field theory. Equations of the form

$$S(q,0) = \text{constant, call it } \sigma$$

[45] He broke new ground in the manner in which he did so, but not in the mere fact that he did so; "quantization conditions" of the form $\oint p\,dq = nh$ began to appear at an early point in the history of the "old quantum mechanics," and in 1916 Karl Schwartzschild remarked that such expressions could in every instance be associated with the "action-angle" variables which are the hallmark of the "Hamilton-Jacobi theory of periodic systems" devised by the celestial mechanic C. E. Delauney in 1846. Paul Ehrenfest was at that same time promoting the view that "Bohr-Sommerfeld quantization conditions" should be associated with certain "adiabatic invariants," and was inspired by Schwartzschild's paper to suggest to Jan Burgers, his thesis student, that it might be possible to develop a "Hamilton-Jacobi theory of adiabatic invariants with quantum mechanical applications." Which Burgers promptly did (though Ehrenfest himself claimed never to have understood the work). The perception that Hamilton-Jacobi theory had important things to say about the quantum world became (at least within a small circle of theorists) progressively more widespread, with the curious result that two of the best accounts of classical Hamilton-Jacobi theory (I refer to Max Born's *The Mechanics of the Atom* (1924) and George Birtwhistle's *The Quantum Theory of the Atom* (1925)) were published with quantum intent just in time to be rendered obsolete by Schrödinger. For further discussion and references, see CLASSICAL MECHANICS (1983), pp. 382–423, especially 418–421.

[46] Which is to say: at such a profoundly *geometrical* level...

describe the initial design of a σ-parametrized population of non-intersecting *surfaces* in configuration space. The Hamilton-Jacobi equation describes the $H(q,p)$-generated *motion* of such surfaces.[47] The fundamental relation

$$\boldsymbol{p}(\boldsymbol{x},t) = \boldsymbol{\nabla} S(\boldsymbol{x},t) \tag{103.1}$$

—more generally

$$p_i(q,t) = \tfrac{\partial}{\partial q^i} S(q,t) \quad : \quad i = 1,2,\ldots,n \tag{103.2}$$

—directs our attention to a dual construction: the population of *curves* which (at time t) thread everywhere normally through the population of S-surfaces. It is through that portal and another—the observation that the functions $p_i(q,t)$ associate p-values with q-values, and serve therefore to inscribe (at time t) a *surface on phase space*, a point-set which is subsequently swept along by the $H(q,p)$-generated phase flow—that one recovers contact with the moving-point imagery of (respectively) the Lagrangian and canonical Hamiltonian formalisms; I will soon have occasion to spell out the meaning this remark in somewhat finer detail.

It is, it seems to me, quite natural to ask (and therefore surprising that, so far as I am aware, no one else has previously thought to ask) whether the Hamilton-Jacobi equation—looked upon as a field equation—can be displayed as an instance of a Lagrangian field theory. Here the Helmholtz conditions (44) come to our aid. Looking to the case (102.1), we have a single field ($N = 1$) written on a spacetime of $m = 2$ dimensions, so the Helmholtz conditions are two in number; they read $\partial G/\partial S_t = 0$ and $\partial G/\partial S_x = 0$; i.e.,

$$1 = 0 \quad \text{and} \quad \tfrac{1}{m} S_x = 0$$

which are absurd: *no Lagrange density exists* which would yield (102.1) as a field equation.

This is an unsettling development, for Hamilton-Jacobi theory is a field theory with a uniquely strong claim to "deep and universal significance;" it seems to me unreasonable on its face that such a theory should be denied access to the rich formal resources of Lagrangian field theory. I interpret this development to be a "call to invention;" we must deepen the channel if we are to get this boat afloat. I describe now two distinct strategies for accomplishing precisely that objective:

FIRST STRATEGY

What might be called the "auxiliary variable trick"[48] is always available, but merits serious attention only in cases where *direct physical significance* can be assigned to the auxiliary field or fields, and to the associated field equations.

[47] It is natural to associate such surfaces with "wavefronts," though—for us as for Huygens—such wavefronts possess no undulatory aspect; we find our pre-quantum mechanical selves contemplating "wavefronts without waves."

[48] See again pp. 39–40.

That we have in hand just such a case will emerge, but only after we have introduced new ideas into the Hamilton-Jacobi formalism itself.[49]

Writing

$$\mathcal{L} = A \cdot \left\{ \tfrac{1}{2m} S_x^2 + U + S_t \right\}$$

we observe that, of dimensional necessity,

$$[A] = \frac{1}{(\text{length})^{\text{spatial dimension}}} = \frac{1}{\text{volume}} = \text{density}$$

and obtain field equations

$$\left\{ \partial_t \frac{\partial}{\partial A_t} + \partial_x \frac{\partial}{\partial A_x} - \frac{\partial}{\partial A} \right\} \mathcal{L} = 0 \quad \text{giving} \qquad \tfrac{1}{2m} S_x^2 + U + S_t = 0$$

$$\left\{ \partial_t \frac{\partial}{\partial S_t} + \partial_x \frac{\partial}{\partial S_x} - \frac{\partial}{\partial S} \right\} \mathcal{L} = 0 \quad \text{giving} \quad \partial_t A + \partial_x \left(\tfrac{1}{m} A S_x \right) = 0$$

The latter equation—which in the 3-dimensional case reads

$$\partial_t A + \boldsymbol{\nabla} \cdot \left(\tfrac{1}{m} A \, \boldsymbol{\nabla} S \right) = 0 \tag{104}$$

—has the form of a *continuity equation*, and therein lies the clue to its meaning. It is to make clear that meaning that we have need of the "new ideas" to which I just referred, and it is to acquire those that I now digress:

Let state points be sprinked onto phase space with initial density given by $P(\boldsymbol{x}, \boldsymbol{p}, 0)$. Those points—transported by the ambient phase flow—have by time t achieved the distribution $P(\boldsymbol{x}, \boldsymbol{p}, t)$. Phase flow is symplectic, therefore

[49] The following discussion has been adapted from material which appears on pp. 435–471 and 489–495 in CLASSICAL MECHANICS (1983). A word concerning my motivation in that work: Feynman's sum-over-paths formulation of quantum mechanics assigns central importance—this is a point first recognized by Pauli and others in 1950/51, a point which seems never to have aroused the interest of Feynman himself—to an object called the *Van Vleck determinant*. That object first appears in a paper ("The correspondence principle in the statistical interpretation of quantum mechanics," PNAS **14**, 178 (1928)) concerned with quantum fundamentals, but is itself entirely classical. My objective was to establish the sense in which the Van Vleck determinant

$$D(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \equiv \det \left\| \frac{\partial^2 S(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)}{\partial x^i \partial x_0^j} \right\|$$

—a construction which has already other work to do within the deeper reaches of Hamilton-Jacobi theory (see p. 259 in V. I. Arnold's *Mathematical Methods of Classical Mechanics* (1989))—acquires a "natural predisposition" to assume its quantum mechanical burden.

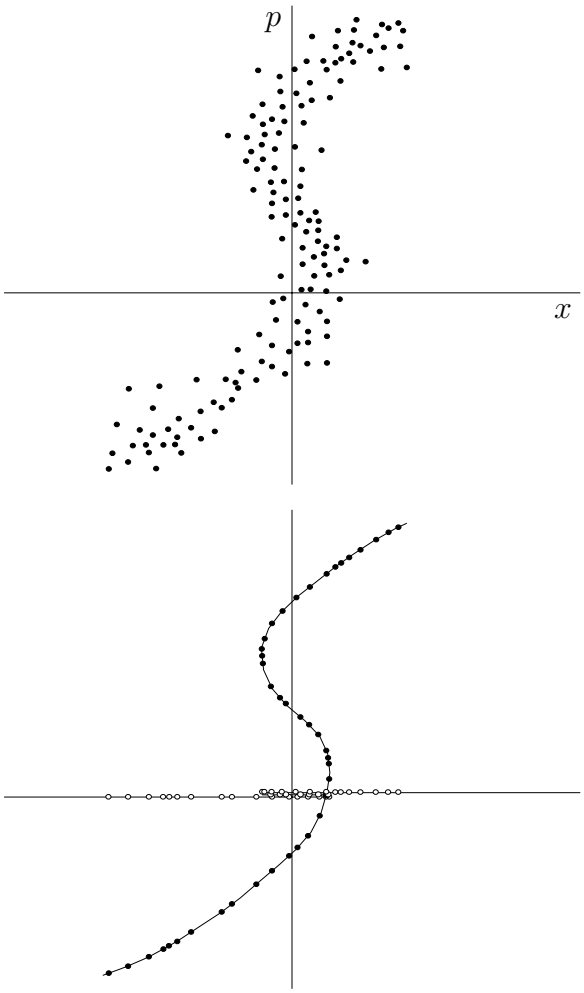


FIGURE 9: *Shown above is an unspecialized distribution $P(x, p, t)$ of state points, such as might be encountered in a statistical mechanical argument. The motion of the distribution is described by (105). In the lower figure the state points have been sprinkled on the surface Σ_t defined by $p(x, t) = S_x(x, t)$. Open circles on the x-axis represent projective images of those points, distributed as described by $A(x, t)$. The distribution in phase space has the specialized structure (106), and $A(x, t)$ satisfies the continuity equation developed in the text. The x-axis is folded to reflect implications of the fact that on a central interval the function $S_x(x, t)$ is triple-valued.*

volume-preserving; therefore the point density in the neighborhood of any co-flowing point is seen to be constant:

$$\begin{aligned}\tfrac{d}{dt}P &= \sum_k \Big\{ \tfrac{\partial P}{\partial x^k}\dot{x}^k + \tfrac{\partial P}{\partial p_k}\dot{p}_k \Big\} + \tfrac{\partial P}{\partial t} \\ &= \sum_k \Big\{ \tfrac{\partial P}{\partial x^k}\tfrac{\partial H}{\partial p_k} - \tfrac{\partial P}{\partial p_k}\tfrac{\partial H}{\partial x^k} \Big\} + \tfrac{\partial P}{\partial t} = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ &\qquad\qquad\qquad\qquad [\,P, H\,] + \tfrac{\partial P}{\partial t} = 0 \end{aligned} \tag{105}$$

which is the upshot of *Liouville's theorem*.[50] In the "pointwise conception" of Hamiltonian mechanics one uses the canonical equations

$$\dot{x}^k = +\tfrac{\partial}{\partial p_k}H \quad \text{and} \quad \dot{p}_k = -\tfrac{\partial}{\partial x^k}H$$

to watch the motion of individual points in phase space, while in the "global conception" one uses (105) to watch the motion of arbitrarily distributed point *populations*. Hamilton-Jacobi gives rise to an "intermediate conception" if one sprinkles state points on the *surface*—call it Σ_t—which (103) serves to inscribe on phase space; to do so is in effect to assume that the distribution function $P(\boldsymbol{x},\boldsymbol{p},t)$ has the specialized structure

$$P(\boldsymbol{x},\boldsymbol{p},t) = A(\boldsymbol{x},t)\cdot\delta\big(\boldsymbol{p} - \boldsymbol{\nabla}S(\boldsymbol{x},t)\big) \tag{106}$$

Here $A(\boldsymbol{x},t)$ provides a "projective" account of *how* state points are distributed on Σ_t; from

$$\int P(\boldsymbol{x},\boldsymbol{p},t)\,dp_1 dp_2 \ldots dp_n = A(\boldsymbol{x},t)\cdot(\text{multiplicity of } \boldsymbol{\nabla}S(\boldsymbol{x},t) \text{ at } \boldsymbol{x})$$

we see that $A(\boldsymbol{x},t)$ is closely related to the *marginal distribution* in $\boldsymbol{x}$, and becomes precisely the marginal distribution at points where $\boldsymbol{\nabla}S(\boldsymbol{x},t)$ is single-valued. Figure 9 provides illustration of the sense in which distributions of type (106) are "specialized," of how $A(\boldsymbol{x},t)$ acquires projective significance, and of why multiplicity matters. Introducing (106) into (105) we obtain[51]

$$\begin{aligned}[\,P, H\,] + \tfrac{\partial P}{\partial t} &= \big\{A\delta(p-S_x)\big\}_x H_p - \big\{A\delta(p-S_x)\big\}_p H_x + \big\{A\delta(p-S_x)\big\}_t \\ &= \big\{A_x\tfrac{p}{m} + A_t\big\}\delta\,(p-S_x) \\ &\qquad - A\,\delta'(p-S_x)\big\{S_{xx}\tfrac{p}{m} + U_x + S_{xt}\big\}\end{aligned}$$

[50] See H. Goldstein, *Classical Mechanics* (2nd edition 1980), §9-8.

[51] I find it notationally convenient (since the argument is a little bit intricate) to restore now my former presumption that $n = 1$. For that same reason, I restrict my explicit remarks to the case $H(p,x) = \frac{1}{2m}p^2 + U(x)$, and use subscripts to denote partial derivatives.

It has, however, been observe by Dirac[52] that $x\,\delta(x) = 0$; by differentiation we have $x\,\delta'(x) = -\delta(x)$ whence $x\,\delta'(x-a) = a\,\delta'(x-a) - \delta(x-a)$, with the aid of which we obtain

$$\begin{aligned}
&= \big\{\tfrac{1}{m}(S_{xx}A + S_xA_x) + A_t\big\}\delta(p - S_x) \\
&\qquad - A\,\delta'(p - S_x)\big\{\tfrac{1}{m}S_xS_{xx} + U_x + S_{xt}\big\} \\
&= \big\{(\tfrac{1}{m}S_xA)_x + A_t\big\}\delta(p - S_x) \\
&\qquad - A\,\delta'(p - S_x)\tfrac{\partial}{\partial x}\big\{\underbrace{\tfrac{1}{2m}S_x^2 + U + S_t}_{0 \quad \text{by the Hamilton-Jacobi equation}}\big\} \\
&= 0 \quad \text{by (105)}
\end{aligned}$$

The implication is that if $P(\boldsymbol{x}, \boldsymbol{p}, t)$ is of the form (106) then Liouville's theorem (105) entails that $A(\boldsymbol{x}, t)$ satisfies the continuity equation

$$A_t + (\tfrac{1}{m}S_xA)_x = 0$$

In several-dimensional cases (by an elaboration of the same argument) we recover (104). The fact that $A(\boldsymbol{x}, t)$ invites interpretation as a spatial point density conforms well, by the way, to an earlier remark relating to its enforced physical dimensionality.

We are brought thus to the conclusion that while the Hamilton-Jacobi theory of the textbooks—a *single-field* theory—resists display as a Lagrangian field theory, the associated theory of "populated" Σ_t-surfaces—a *two-field theory*, with (in the simplest instance) field equations

$$\left.\begin{aligned}
\tfrac{1}{2m}\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}S + U + \tfrac{\partial}{\partial t}S = 0 \\
\tfrac{\partial}{\partial t}A + \boldsymbol{\nabla}\cdot(\tfrac{1}{m}A\boldsymbol{\nabla}S) = 0
\end{aligned}\right\} \tag{107}$$

—does admit of such display, and derives in fact from a very simple Lagrangian:

$$\mathcal{L} = A \cdot \big\{\tfrac{1}{2m}\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}S + U + \tfrac{\partial}{\partial t}S\big\} \tag{108}$$

SECOND STRATEGY

In 1979, when I first considered the problem of bringing Hamilton-Jacobi theory within the rubric of of Lagrangian field theory,[53] I was led—with the "integrating factor trick" (see again p. 34) and the "auxiliary field trick"[48] jointly in mind, and "after a bit of exploratory tinkering"—to construct

$$\mathcal{L}_a = e^{S/a}\Big\{\tfrac{a}{2m}S_xB_x + aB_t - UB\Big\} \tag{109}$$

[52] *Principles of Quantum Mechanics* (4th edition, 1958), (7) in §15.

[53] See "A sense in which classical mechanics 'quantizes itself':" notes for a seminar presented 21 January 1980 at Portland State University—appended to CLASSICAL FIELD THEORY (1979).

where a is a constant of (for the moment) arbitrary value but the *dimensionality of action* (made necessary because the exponentiated S carries that dimension) and the auxiliary field B is co-dimensional with the previously-encountered auxiliary field A. The Lagrangian (109) yields field equations

$$\Big\{\partial_t \frac{\partial}{\partial B_t} + \partial_x \frac{\partial}{\partial B_x} - \frac{\partial}{\partial B}\Big\}\mathcal{L}_a = e^{S/a}\Big\{\tfrac{1}{2m}S_x^2 + U + \tfrac{a}{2m}S_{xx} + S_t\Big\} = 0$$
$$\Big\{\partial_t \frac{\partial}{\partial S_t} + \partial_x \frac{\partial}{\partial S_x} - \frac{\partial}{\partial S}\Big\}\mathcal{L}_a = e^{S/a}\Big\{\tfrac{a}{2m}B_{xx} - B_t + \tfrac{1}{a}UB\Big\} = 0$$

which entail

$$\tfrac{1}{2m}S_x^2 + U + \tfrac{a}{2m}S_{xx} + S_t = 0 \tag{110.1}$$
$$\tfrac{a^2}{2m}B_{xx} + UB - aB_t = 0 \tag{110.2}$$

The latter equation, though formally interesting on account of its linearity, appears at present to have the status of a mere artifact, and—particularly since uncoupled to its companion—to have no direct claim to our attention. The former equation contains an "unwanted S_{xx}-term," but that term disappears in the limit $a \downarrow 0$. Evidently the Hamilton-Jacobi equation can be harvested as fruit of the somewhat odd procedure

$$\lim_{a\downarrow 0}\left\{\partial_t \frac{\partial}{\partial B_t} + \partial_x \frac{\partial}{\partial B_x} - \frac{\partial}{\partial B}\right\}\mathcal{L}_a = 0 \tag{111}$$

Formulation of the higher-dimensional analog of this result poses no problem.

The *deus ex machina* is never a welcome participant in scientific discourse. Good theories, like good machines, accomplish their work with the least number of moving parts. We have learned to hold in contempt theories which rely upon the assistance of crutches, which contain "unused parts," and our contempt is the higher the more "fundamental" the theory purports to be. We are motivated by this train of thought to inquire more closely into possibly deeper significance of the auxiliary field B. Such a program entails that we *suspend the limit procedure* $a \downarrow 0$ and *take seriously the two-field theory implicit in* $\mathcal{L}_a$; it entails that we divert our attention from the (solitary) Hamilton-Jacobi equation itself and look with focused attention to the (uncoupled) *pair* of field equations (111).

How to proceed? The question vitually answers itself the minute one notices that adjustment of what one understands to be the "field function"

$$S \longrightarrow \Psi \equiv e^{S/a}$$

entails

$$a^2\Psi_{xx} = e^{S/a}\big\{S_x^2 + aS_{xx}\big\}$$
$$a\Psi_t = e^{S/a}S_t$$

and permits the Lagrange equation $e^{S/a}$(110.1) to be notated

$$\tfrac{a^2}{2m}\Psi_{xx} + U\Psi + a\Psi_t = 0 \tag{112}$$

Remarkably, we have managed (unwittingly) to *linearize* (110.1), and have achieved an equation which—except for the final sign—mimics the structure of (110.2). That mimicry can be made precise by a very simple device: assume a to be *imaginary*, writing

$$a = -i\hbar$$

where $\hbar$ is for the moment not to be confused with Planck's constant: it is a real variable of arbitrary value, and dimensionality $[\hbar]$ = action. Then (112) reads

$$-\tfrac{\hbar^2}{2m}\Psi_{xx} + U\Psi - i\hbar\Psi_t = 0 \tag{113.1}$$

which by conjugation—note that

$$\Psi = e^{\frac{i}{\hbar}S} \quad \text{has become a } \textit{complex-valued} \text{ field function}$$

—assumes precisely the form

$$-\tfrac{\hbar^2}{2m}\Psi^*_{xx} + U\Psi^* + i\hbar\Psi^*_t = 0$$

of (110.2), which now reads[54]

$$-\tfrac{\hbar^2}{2m}B_{xx} + UB \ \ + i\hbar B_t = 0 \tag{113.2}$$

and can be considered to have arisen (together with (113.1)) from this notational variant of (109):

$$\mathcal{L}_\hbar = -\tfrac{\hbar^2}{2m}\Psi_x B_x - i\hbar\Psi B_t - \Psi UB \tag{114}$$

Beyond this point there are several ways to proceed: it is tempting to *identify* (113.2) with the conjugate of (113.1), but dimensional circumstances stand in the way:

$$\Psi\text{—whence also } \Psi^*\text{—are dimensionless, while } [B] = (\text{volume})^{-1}$$

We might, at (114), circumvent this problem by making formal replacements

$$\left.\begin{array}{l}\Psi \hookrightarrow \psi \\ B \hookrightarrow \psi^*\end{array}\right\} \quad \text{with} \quad [\psi] = [\psi^*] = (\text{volume})^{-\frac{1}{2}}$$

We then obtain (after a physically insignificant overall sign reversal)

$$\begin{aligned}\mathcal{L} &= \tfrac{\hbar^2}{2m}\psi^*_x\psi_x + i\hbar\,\psi^*_t\psi + \psi^*U\psi \\ &\quad\downarrow \\ &= \tfrac{\hbar^2}{2m}\psi^*_x\psi_x + \tfrac{1}{2}i\hbar\,(\psi^*_t\psi - \psi^*\psi_t) + \psi^*U\psi\end{aligned} \tag{115}$$

where the point of the final manipulation has been (by gauge transformation) to restore the manifest *reality* of $\mathcal{L}$.

[54] Note that the following equation forces the complexification also of the B-field.

Continuing in the rhetorical pretense that we are "classical physicists who don't already know quantum mechanics," we might argue that the theory latent in (115) is "so pretty that it might be right." But what to do about $\hbar$? Most systems do not supply constants/parameters sufficient to permit assembly of a "natural action." To lend the theory a universality commensurate with its beauty we would be forced to assign $\hbar$ the status of a "constant of Nature" ... and ultimately the observational value

$$\hbar = 1.054592 \times 10^{-27}\text{erg-seconds}$$

We have at (115) recovered precisely the Lagrangian which in discussion subsequent to (76) was found to give rise (with the assistance of Noether's theorem) to all formal aspects of quantum-mechanics-according-to-Schrödinger, though not to enforce the *interpretive* aspects of the latter theory.[55] The classical mechanics of a particle—when written as a (Hamilton-Jacobi) field theory and forced (by these means) into conformity with the general rubric of Lagrangian field theory—has, in effect, "quantized itself." At what point did classical mechanics become something other than it was? Several distinct actions—each seemingly slight, and formally natural in itself—contributed to this development:

- We left classical mechanics behind when we agreed to suspend the limiting process $a \downarrow 0$;
- We went further afield when at $a \to -i\hbar$ we complexified the parameter a, which entailed complexification also of $\Psi = e^{S/a}$ and of B; a 2-field theory had at that point become a 4-field theory.

By that point we had achieved a theory which was *neither* classical mechanics nor quantum mechanics; to achieve the latter (a 2-field theory) we had finally to make the replacements $\Psi \hookrightarrow \psi, B \hookrightarrow \psi^*$.

I now reverse the trend of the argument; taking (115) as my point of departure, I use field-theoretic methods to proceed back again toward the Hamilton-Jacobi theory from which we came:

$$\text{Hamilton-Jacobi theory} \longleftarrow \text{quantum mechanics}$$

We make essential use of the polar representation

$$\psi = \mathsf{R}\exp\left\{\tfrac{i}{\hbar}\mathsf{S}\right\} \tag{116}$$

and will find that most of the work has already been done; the exercise will, however, serve to establish a bridge of the form

[55] It is from those "interpretive aspects"—due to Born—that we acquire rationale for the imposition of *boundary conditions* upon ψ, and also the theory of measurement which becomes the source of (amongst other things) our *initial value* assertions.

FIRST STRATEGY $\longleftrightarrow$ SECOND STRATEGY

and to illuminate the theoretical placement of the so-called "de Broglie-Bohm interpretation of quantum mechanics."

In the discussion culminating in (91) we found it to be (by Noether's theorem) an implication of the *phase-invariance* of $\mathcal{L}$ that

$$\partial_t P + \boldsymbol{\nabla}\cdot\boldsymbol{P} = 0 \quad : \quad \left\{ \begin{array}{l} \text{standardly interpreted to} \\ \text{express "probability conservation"} \end{array} \right.$$

where in polar notation (see again (90)) $P = \mathsf{R}^2$ and $\boldsymbol{P} \equiv \frac{1}{m}\mathsf{R}^2\boldsymbol{\nabla}\mathsf{S}$. So we have

$$\downarrow$$

$$\partial_t(\mathsf{R}^2) + \boldsymbol{\nabla}\cdot(\tfrac{1}{m}\mathsf{R}^2\boldsymbol{\nabla}\mathsf{S}) = 0 \tag{117.1}$$

which bears a striking resemblance to (104), to which it would revert upon substitutional transformation

$$\mathsf{R}^2 \hookrightarrow A \quad \text{and} \quad \mathsf{S} \hookrightarrow S$$

We are, in the light of this observation, not surprised to recall from (78) that introduction of (116) into the higher-dimensional generalization of (115) yields a Lagrangian

$$\mathcal{L} = \mathsf{R}^2 \cdot \big[\mathsf{S}_t + \tfrac{1}{2m}\boldsymbol{\nabla}\mathsf{S}\cdot\boldsymbol{\nabla}\mathsf{S} + U\big] + \tfrac{\hbar^2}{2m}\mathsf{R}_x^2$$

which has *except for the "dangling term"* precisely the structure of (108), and gives rise to field equations (117.1) and

$$2\mathsf{R}\big[\tfrac{1}{2m}\boldsymbol{\nabla}\mathsf{S}\cdot\boldsymbol{\nabla}\mathsf{S} + U + \mathsf{S}_t\big] = \tfrac{\hbar^2}{m}\nabla^2\mathsf{R}$$

$$\downarrow$$

$$\tfrac{1}{2m}\boldsymbol{\nabla}\mathsf{S}\cdot\boldsymbol{\nabla}\mathsf{S} + \big\{U - \tfrac{\hbar^2}{2m}\tfrac{1}{\mathsf{R}}\nabla^2\mathsf{R}\big\} + \tfrac{\partial}{\partial t}\mathsf{S} = 0 \tag{117.2}$$

which—except for a solitary extra term of order $O(\hbar^2)$—possess precisely the structure of the Hamilton-Jacobi system (107).

It will be appreciated that equations (117)—their classical appearance notwithstanding—refer to the *quantum* dynamics of a particle; taken together, they are equivalent to (simply a notational variant of) the Schrödinger equations

$$-\tfrac{\hbar^2}{2m}\nabla^2\psi + U\psi = i\hbar\partial_t\psi \quad \text{and its complex conjugate} \tag{118}$$

The classical dynamics of a particle (Hamilton-Jacobi formulation: (107)) is recovered if, in the spirit of several-dimensional WKB theory,[56] one introduces

$$\begin{aligned} \mathsf{S} &= \ S \ + \hbar\mathsf{S}_1 + \hbar^2\mathsf{S}_2 + \cdots \\ \mathsf{R} &= \sqrt{A} + \hbar\mathsf{R}_1 + \hbar^2\mathsf{R}_2 + \cdots \end{aligned}$$

into (117) and retains only the terms of 0^{th} order.

[56] See QUANTUM MECHANICS (1967), Chapter I, p. 88–106.

Looking to the "field-theoretic reconstruction of quantum mechanics," as outlined above, we see that the theory acquired its undulatory aspects—became literally a "wave mechanics"—from the i introduced at $a \to -i\hbar$.[57] Only with those undulatory aspects in place does it become possible (with de Broglie) to entertain associations of the form

$$\text{wavelength} \sim \hbar/\text{momentum}$$
$$\text{frequency} \sim \text{energy}/\hbar$$

Surfaces of constant S admit literally of interpretation as "wavefronts: surfaces of constant phase;" in the limit $\hbar \downarrow 0$ they become surfaces of constant S, the spooky "wavefronts without waves" which Hamilton-Jacobi theory leads one to contemplate.[47]

The connection between "the field theory we call quantum mechanics" and "the field theory we call Hamilton-Jacobi theory" is so richly deep that it is encountered at every turn, and in many guises, whenever one looks either to "the quantization problem" or to its obverse, "the classical limit (correspondence) problem." Schrödinger himself worked from one perception of that connection, Feynman from another. A particularly beautiful account of the intrusion of Hamilton-Jacobi theory into quantum mechanics was devised by Whittaker in some early work[58] which attracted little attention at the time and is today almost forgotten, though in several respects it was anticipatory both of Feynman and of Schwinger. Diverse beasts drink at the same waterhole. On this occasion I will digress to discuss only one of those:

In 1952 David Bohm published the first[59] of series of papers in which he took the Schrödinger equation (117≡118) as his point of departure, but imported from classical mechanics an equation

$$\boldsymbol{p} = \boldsymbol{\nabla}\mathsf{S} \tag{119}$$

imitative of (103). By the latter strategem he made available to quantum mechanics a physical image—populated surface moving in phase space—basic to classical Hamilton-Jacobi theory (see again Figure 9). Given a solution ψ of (118)—and therefore already in possession of all the testable statements his theory is capable of providing—Bohm proceeds to "play classical mechanics"

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[57] This i is responsible also for the fact that we had to make a "physically insignificant. . . sign reversal" in order to achieve (115). Such a reversal sends $p = \partial\mathcal{L}/\partial q \to -p$, and will be "physically insignificant" only if the physics in question is *time-reversal invariant*.

[58] E.T.Whittaker, "On Hamilton's principal function in quantum mechanics," Proc. Roy. Soc. Edinburgh **61A**, 1 (1941). For an exposition of Whittaker's work see "The quantum mechanical Hamilton-Jacobi equation" in QUANTUM MECHANICS (1967), Chapter 3, pp. 68–83.

[59] "A suggested interpretation of the quantum theory in terms of 'hidden variables,' Parts I & II," Phys. Rev. **85**, 166 & 180. These papers are reprinted in J.A.Wheeler & W.H.Zurek, *Quantum Theory and Measurement* (1983).

in order to gain what he imagines to be a philosophical advantage. By the phrase "play classical mechanics" I mean this: he elects to construe (117) as a specialized mutant of (107). "Specialized" in this regard: to the "point density" function A he assigns the ψ-dictated structure

$$A \xrightarrow[\text{Bohm's specialization}]{} \psi^*\psi = \mathsf{R}^2$$

And "mutant" in this:

$$U \xrightarrow[\text{Bohm's adjustment}]{} U+Q$$
$$Q \equiv -\tfrac{\hbar^2}{2m}\tfrac{1}{\mathsf{R}}\nabla^2\mathsf{R} = -\tfrac{\hbar^2}{2m}\tfrac{1}{2A}\big[\nabla^2 A - \tfrac{1}{2A}\boldsymbol{\nabla}A{\cdot}\boldsymbol{\nabla}A\big]$$
$$\equiv \text{so-called "quantum potential"}$$

In Bohm's quantum-adjusted version of classical mechanics the populated surface Σ_t inscribed on phase space by (119) does *not*—owing to the presence of the "quantum potential"—move classically (except in the limit $\hbar \downarrow 0$), and in one respect its motion is qualitatively quite *un*classical: the motion of Σ_t *depends upon the density A with which state points have been sprinkled upon it*.[60] That detail notwithstanding, Bohm promotes the view that the particle trajectories which emerge from the system

$$\left.\begin{aligned} -\tfrac{\hbar^2}{2m}\nabla^2\psi + U\psi &= i\hbar\partial_t\psi \\ \tfrac{1}{2m}\boldsymbol{\nabla}\mathsf{S}{\cdot}\boldsymbol{\nabla}\mathsf{S} + \big\{U+Q\big\} + \tfrac{\partial}{\partial t}\mathsf{S} &= 0 \\ Q \text{ constructed from } \psi, \text{ as explained above} & \\ \boldsymbol{p} = \boldsymbol{\nabla}\mathsf{S} & \end{aligned}\right\} \qquad (120)$$

are as "objectively real" as those which emerge from the Hamilton-Jacobi system (107), and that ψ provides representation of a force field as objectively real as (say) the electromagnetic field. A detailed account of the resulting "causal interpretation of quantum mechanics" (the outlines of which were anticipated already by de Broglie in 1932) can be found in a recent monograph by Peter Holland,[61] and a broad selection of more narrowly focused essays—some by major figures (Roger Penrose, Berhard d'Espagnat, Richard Feynman, Y. Aharonov, J. S. Bell and others), some by assorted philosophers and psychiatrists—has been edited by B. J. Hiley & F. David Peat.[62]

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[60] Bohm himself might have taken exception to my image of "state points sprinkled on Σ_t. He imagined himself to be describing the motion of a *single* state point (or particle), and the intrusion of A into the theory to be "merely a consequence of our [unavoidable and irreducible] ignorance or the precise initial conditions of the particle." In my view the distinction is mainly one of language and emphasis, though it is for Bohm the source of his allusion to "hidden variables."

[61] *The Quantum Theory of Motion* (1993). Holland's §2.6—entitled "Classical mechanics as a field theory" (see also his equation (8.14.19))—bears a striking similarity to material presented in recent pages, though his point of departure and intended destination are both quite different from my own.

[62] *Quantum Implications: Essays in Horour of David Bohm* (1987).

The question "Is quantum mechanics complete?" was debated to a virtual standstill by Bohr and Einstein during the years 1925–31,[63] and brought (or so it was imagined) to a kind of conclusion (in the affirmative) by von Neumann, whose "impossibility proof"[64] appeared in 1932. That the question is today more alive than ever, and exploration of the "foundations of quantum mechanics" has become a cottage industry, is due in substantial part to Bohm. Not that Bohm attracted (except from fringe elements) many converts to his views: most physicists were no more inclined than I am to optimism that our understanding of quantum mechanics can be deepened in any significant or useful way by the addition of elements which can be defended/rebutted only by philosophical debate, and many found technical fault with Bohm's proposal.[65] Bohm's 1952 papers did, however, draw attention to the fact that von Neumann's assumptions were susceptible to criticism (and, latently, to circumvention), and did lend encouragement to those who sought—for whatever reason—to escape the dogma of the "Copenhagen interpretation;" they did, in short, inspire thought. Thus J. S. Bell, who at one point wrote[66] that "Bohm's... papers were for me a revelation. ... I have always felt that people who have not grasped the ideas of those papers... are handicapped in any discussion of the meaning of quantum mechanics." Y. Aharonov remarks that Bohm's theory is "often accused of artificiality and inelegance, and doubtless it is guilty of both," but continues: "But to make such accusations, and to leave it at that, is to entirely miss the point. What Bohm was after... was not elegance and not naturalness; Bohm's intentions were simply to produce a theory which, *whatever* its other characteristics, had *logically clear foundations*..."[67] Quoting again from Bell: "It is easy to find good reasons for disliking the de Broglie–Bohm picture. Neither de Broglie nor Bohm liked it very much; for both of them it was only a point of departure. Einstein also did not like it very much.

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[63] See P. A. Schilpp, *Albert Einstein: Philosopher-Scientist* (1951), Chapter 7; Abraham Pais, *'Subtle is the Lord...': The Science and the Life of Albert Einstein*, Chapter 25 or Chapter 5 of Jammer (cited below). It is interesting to recall in this connection that the title of the famous EPR paper, which appeared somewhat later (A. Einstein, B. Podolsky & N. Rosen, Phys. Rev. **47**, 777 (1935)), is "Can the quantum mechanical description of Nature be considered complete?"

[64] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (1932), Chapter 4, §§1 & 2.

[65] For a wonderfully detailed account of the issues surrounding Bohm's work and its reception, see Max Jammer, *The Philosophy of Quantum Mechanics: The Interpretations of Quantum Mechanics in Historical Perspective* (1974), Chapter 7, especially §7.5.

[66] "Beables for quantum field theory," *Speakable and unspeakable in quantum mechanics* (1987), reprinted in Hiley & Peat.

[67] The quotation is taken from "The issue of retrodiction in Bohm's theory," which appears in Hiley & Peat. The "Aharanov-Bohm effect" was first described in "Significance of electromagnetic potentials in quantum theory," Phys. Rev. **115**, 485 (1959).

He found it 'too cheap,' although, as Born remarked, 'it was quite in line with his own ideas.' "[68]

I feel I owe my reader an explanation: Why—in a work ostensibly concerned with the classical theory of fields—have I allowed myself the indulgence of the preceding digression? The answer resides in the collision of two circumstances: We found it to be the case that the classical mechanics of a particle, when approached field theoretically from just the right angle, very nearly "quantizes itself," but that the line of argument in question only hints at—and certainly does not enforce—the *interpretive* statements standard to quantum theory. Oz Bonfim—my colleague, and the occupant of the office next to mine—has in recent years been exploring this question: Can the "orbital concreteness" which the Bohm theory introduces into quantum theory be exploited to lend sharper meaning to the notion of "quantum chaos"? It became therefore natural to ask: Does the field theoretic quantization procedure lead to a formalism as "naturally predisposed" to Bohm's "causal interpretation" as to the standard (Bohr/Born) interpretation? Had the answer been in the affirmative, I would have looked upon Bohm's creation with more enthusiasm (which is to say: with less dubiousness) than has been my habit. But I have come to the conclusion that appropriation of the equation

$$\boldsymbol{p} = \boldsymbol{\nabla}\mathsf{S}$$

—an act which lies at the heart of Bohm's program—is field theoretically unmotivated; the equation is a gratuitous import which, since it stands with one leg planted in a formalism dominated by the symplectic group and the other in a formalism dominated by the unitary group, leads to a fantasy at risk of becoming transformation-theoretically unstuck unless managed carefully.

I conclude with Bohm-inspired remarks which culminate in formulation of an open question which seems to me to be of independent field theoretic interest. From (116) it follows (recall (90)) that

$$\begin{aligned}
\mathsf{R}^2 &= \psi^*\psi \\
&= \text{probability density} \\
\boldsymbol{\nabla}\mathsf{S} &= \hbar\boldsymbol{\nabla}\arctan\Big\{i\,\frac{\psi^* - \psi}{\psi^* + \psi}\Big\} \\
&= \tfrac{1}{2}i\hbar\,\frac{\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi}{\psi^*\psi} \\
&= m\cdot\frac{\text{probability current}}{\text{probability density}}
\end{aligned}$$

This information could have been extracted directly from (117.1), but I find the present argument amusing. The immediate point, however, is this: we have

[68] See §3 in "On the impossible pilot wave," which appears as Essay 17 in the collection cited previously.[66] Also the appendix to the paper reprinted as Essay 10: "Einstein–Podolsky–Rosen experiments."

shown (104) to originate in (106); we have, in other words, shown that (117.1) would follow from

$$\left.\begin{aligned} P(\boldsymbol{x},\boldsymbol{p},t) &= \mathsf{R}^2(\boldsymbol{x},t)\cdot\delta(\boldsymbol{p}-\boldsymbol{\nabla}\mathsf{S}) \\ &= (\text{probability density})\cdot\delta\Big(\boldsymbol{p}-m\,\frac{\text{probability current}}{\text{probability density}}\Big) \\ &= \psi^*\psi\cdot\delta\Big(\boldsymbol{p}-\tfrac{1}{2}i\hbar\,\frac{\psi\boldsymbol{\nabla}\psi^*-\psi^*\boldsymbol{\nabla}\psi}{\psi^*\psi}\Big) \\ &= \psi^*\psi\cdot\delta\Big(\boldsymbol{p}-\tfrac{1}{2}i\hbar\big[\boldsymbol{\nabla}\log\psi^*-\boldsymbol{\nabla}\log\psi\big]\Big) \end{aligned}\right\} \tag{121}$$

which provide alternative formulations of Bohm's fundamental assumption: he takes the "pilot wave" ψ to be simultaneously responsible for

- the instantaneous design of the surface Σ_t, and
- how Σ_t is "populated"

with the result that in his theory "motion depends upon the population" (through, he would say, the "quantum potential"). There are, however, other ways than (121) to get from $\psi(\boldsymbol{x})$ into phase space. One of those—fundamental to the so-called "phase space formulation of quantum mechanics"—has already been mentioned;[40] I refer to the process

$$\psi(x) \longrightarrow P(x,p) = \tfrac{2}{\hbar}\int \psi^*(x+\xi)e^{2\frac{i}{\hbar}p\xi}\psi(x-\xi)\,d\xi \tag{122}$$

devised by E. P. Wigner and L. Szilard sometime prior to 1932. The Bohm distribution (121) and the Wigner distribution (122), if extracted from identical ψ functions, yield identical marginal distributions

$$\int P(x,p)\,dp = |\psi(x)|^2$$

which, in their separate ways, they launch identically into motion.[69] But when plotted, $P_{\text{Wigner}}(x,p)$ and $P_{\text{Bohm}}(x,p)$ could hardly be more different; the former (if we make allowances for the fact that it can assume negative values) resembles Figure 9^{upper}, while the latter (except for the fact that it can never display multivaluedness) resembles Figure 9_{lower}. Holland,[70] after remarking that "there have been many attempts to introduce phase-space-type structures into quantum mechanics," discusses only one—the formalism based upon the Wigner distribution, which he promptly dismisses on the grounds that it does not conform to Bohm's conception of the world ("does not appear to provide a suitable language for a causal representation of quantum phenomena"). The "phase space formulation of quantum mechanics," as elaborated by J.E. Moyal,[71]

[69] On this evidence we conclude that while probability conservation (91) permits, it *does not enforce* Bohm's structural assumption (121).

[70] See §8.4.3 in the monograph[61] cited previously.

[71] "Quantum mechanics as a statistical theory," Proc. Camb. Phil. Soc. **45**, 99 (1949).

is, in fact, *precisely equivalent* to orthodox quantum mechanics, of which it is an informative reorganization, but into which it imports no fundamentally new physical concepts (no gratuitous innovations subject defense/refutation only by philosophers); therein, for many, lies one of its strengths.[72]

The role of the Schrödinger equation is (in the one-dimensional case) taken over within the phase space formalism by the equation[73]

$$\tfrac{\partial}{\partial t}P = \tfrac{2}{\hbar}\sin\Big\{\tfrac{\hbar}{2}\big[\big(\tfrac{\partial}{\partial x}\big)_H\big(\tfrac{\partial}{\partial p}\big)_P - \big(\tfrac{\partial}{\partial x}\big)_P\big(\tfrac{\partial}{\partial p}\big)_H\big)\big]\Big\}HP \tag{123.1}$$

which gives back Liouville's equation (105) in the limit $\hbar \downarrow 0$. Expanding the sine, we find (123.1) to be a partial differential equation of *infinite order* (though in special cases the power series may truncate). The question which I take this opportunity to pose—but will not attempt to resolve—is this: Can (123.1) be rendered into the language of Lagrangian field theory? The question acquires urgency for the same reason as motivated us when we looked at the beginning of this discussion to the Hamilton-Jacobi equation: the "quantum Liouville equation" (123.1) can plausibly be claimed to be "fundamental." And it acquires formal interest from the circumstance that if a suitable Lagrangian could be found, it would have necessarily the form

$$\mathfrak{L}(P, \partial P, \partial\partial P, \partial\partial\partial P, \cdots)$$

The field equation (123.1) can alternatively be cast as an integral equation

$$\tfrac{\partial}{\partial t}P(x,p,t) = \iint \mathcal{K}(x,p;x_0,p_0)P(x_0,p_0,t)\,dx_0dp_0 \tag{123.2}$$

$$\mathcal{K}(x,p;x_0,p_0) \text{ suitably defined}^{73}$$

which speaks on its face of *non-local* field theory. Whether we were to proceed from (123.1) or from (123.2), we would be obligated to undertake at the outset a major *enlargement of Lagrangian field theory*, as I have presented it.

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[72] For an excellent brief account of the phase space formalism, see pp. 422–425 in Jammer.[65]

[73] For detailed discussion see QUANTUM MECHANICS (1967), Chapter 3, p.110.