Advances in the theory of box integrals

D.H. Bailey[*]   J.M. Borwein[†]   R.E. Crandall[‡]

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Abstract.   Box integrals—expectations $\langle |\vec{r}|^s \rangle$ or $\langle |\vec{r} - \vec{q}|^s \rangle$ over the unit n-cube (or n-box)—have over three decades been occasionally given closed forms for isolated n, s. By employing experimental mathematics together with a new, global analytic strategy, we prove that for $n \le 4$ dimensions the box integrals are for any integer s hypergeometrically closed in a sense we clarify herein. We supply a compendium of exemplary closed forms that naturally arise algorithmically from this theory. These machinations lead also to the resolution of some difficult definite integrals.

1 Preliminaries

We define *box integrals*[1] for dimension n as expectations of $|\vec{r}|^s$, $|\vec{r} - \vec{q}|^s$ with the relevant vectors chosen randomly, independent, equidistributed over the unit n-cube[2] [13]:

$$
\begin{aligned}
B_n(s) &:= \int_{\vec{r} \in [0,1]^n} r^s \, \mathcal{D}\vec{r} \\
&= \int_0^1 \cdots \int_0^1 \left(r_1^2 + \cdots + r_n^2 \right)^{s/2} \, dr_1 \cdots dr_n, \tag{1} \\
\Delta_n(s) &:= \int_{\vec{r}, \vec{q} \in [0,1]^n} |\vec{r} - \vec{q}|^s \, \mathcal{D}\vec{r} \, \mathcal{D}\vec{q} \\
&= \int_0^1 \cdots \int_0^1 \left((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2 \right)^{s/2} \, dr_1 \cdots dr_n \, dq_1 \cdots dq_n. \tag{2}
\end{aligned}
$$

Here, $\mathcal{D}\vec{r}$ denotes simply the volume element $dr_1 dr_2 \cdots$. As explained in a previous treatment [5], there are physical interpretations:

1. $B_n(1)$ is the expected distance of a random point from the origin (or from any fixed vertex) of the n-cube;

2. $\Delta_n(1)$ is the expected distance *between* two random points of the n-cube;

3. $B_n(-n+2)$ is the expected electrostatic potential in an n-cube whose origin has a unit charge;[3]

4. $\Delta_n(-n+2)$ is the expected electrostatic energy between two points in a uniform cube of charged "jellium."

Note that the definitions show immediately that both $\Delta_n(2m)$ and $B_n(2m)$ are rational when m, n are natural numbers. A pivotal, original treatment on box integrals is the 1976 work of Anderssen et al, [1]. There have been interesting modern treatments of the B_n and related integrals, as in [7], [11, p.208], [26], [24]. Related material is also to be found in [16, 25].

There are other similar entities such as the expected distance between points on distinct sides of a cube or hypercube investigated in [11, §1.7] or [7]. We remark that $B_3(1)$ is also known as the *Robbins constant*, after [21].

It is worth noting that the methods of the present paper actually have practical application—for example in biology, of all places. In the work [14] these new box-integral

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[1]Not to be confused with "box integrals" of particle physics.

[2]We refer to "box integrals" but use the more standard term "cube" or "n-cube" for the domain hypercube.

[3]Such statements presume that electrostatic potential in n dimensions is $V(r) = 1/r^{n-2}$, and say $\log r$ for $n = 2$; the main issue being that negative powers of r can also have physical meaning.

evaluations are being used as one measure of "how random?" is a point cloud. That is, one test of whether brain synapses are sufficiently randomly distributed involves comparison of empirical values $\langle |\vec{r} - \vec{q}|^s \rangle$, where $\vec{r}, \vec{q}$ run over synapse positions, against our present theoretical tabulations of $\Delta_3(s)$.

2 Quadrature formulae for all complex powers

As in our previous treatment [5], we define two key functions as

$$b(u) \; := \; \int_0^1 e^{-u^2 x^2} \; dx \; = \; \frac{\sqrt{\pi} \; \mathrm{erf}(u)}{2u}, \tag{3}$$

$$d(u) \; := \; \int_0^1 \int_0^1 e^{-u^2 (x-y)^2} \; dx \; dy \; = \; \frac{-1 + e^{-u^2} + \sqrt{\pi} \; u \; \mathrm{erf}(u)}{u^2}. \tag{4}$$

These functions may be used as integration kernels, in the following way. Writing

$$R^s \; = \; \frac{1}{\Gamma(-s/2)} \int_0^\infty t^{-s/2-1} e^{-tR^2} \; dt, \tag{5}$$

valid for $\Re(s) < 0$, we can integrate formally over R (being a radius, or a separation) in a relevant region, to obtain

$$B_n(s) \; = \; \frac{2}{\Gamma(-s/2)} \int_0^\infty u^{-s-1} b^n(u) \; du, \tag{6}$$

$$\Delta_n(s) \; = \; \frac{2}{\Gamma(-s/2)} \int_0^\infty u^{-s-1} d^n(u) \; du, \tag{7}$$

both representations being valid for a *reduced* range of complex s, namely the range $\Re(s) \in (-n, 0)$. This domain of convergence for the integrals can be inferred from the large-u asymptotic behaviors, see e.g., [2],

$$b(u) = \frac{1}{u} + O\left(e^{-u^2}\right), \tag{8}$$

$$d(u) \; \sim \; \frac{\sqrt{\pi}}{u} - \frac{1}{u^2} + O\left(e^{-u^2}\right), \tag{9}$$

so that having integrand factors b^n, d^n respectively allows integral convergence for the stated range $\Re(s) \in (-n, 0)$.

What can we say about analytic continuation in s? We can extend such previous results to *all* complex s, by paying deeper attention to integrand structure. First, we can extend the region for negative $\Re(s)$ arbitrarily, by finding superpositions of terms $b(ku)$ or $d(ku)$, for integers k, such that the n-th powers of the asymptotic forms (8, 9) are genuinely of exponential decay.

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Second, for positive $\Re(s)$ and a positive integer K, we may employ the integral identity

$$D(\rho, K) \;:=\; \int_0^\infty t^{-\rho-1}\left(1 - e^{-tu}\right)^K \; du = \Gamma(-\rho)\sum_{j=1}^{K}\binom{K}{j}(-1)^j j^\rho. \qquad (10)$$

from which it is straightforward to derive integral representations for all s in the open right half-plane except for the positive even integers.

Theorem 1 [Quadrature formulae for general complex s] *In the region with $\Re(s) < 0$ we have*

$$B_n(s) \;=\; \frac{2}{1 - 2^{n+s}}\frac{1}{\Gamma(-s/2)}\int_0^\infty u^{-s-1}\left(b^n(u) - 2^n b^n(2u)\right)\; du. \qquad (11)$$

For $\Delta_n(s)$ in the same s-region, let $A_1 := 1$ and consider the unique solution $(A_2, \ldots, A_{n+2})$ to the linear system

$$0 = \sum_{k=1}^{n+2}\frac{A_k}{k^q}, \qquad q = 0, 1, 2, \ldots, n. \qquad (12)$$

Then

$$\Delta_n(s) \;=\; \frac{2}{\Gamma(-s/2)}\left(\sum_{k=1}^{n+2}A_k k^{n+s}\right)^{-1}\int_0^\infty u^{-s-1}\left(\sum_{k=1}^{n+2}A_k k^n d^n(ku)\right)\; du. \qquad (13)$$

Next, consider the region $\Re(s) \in (0, 2K)$ with K a positive integer and s not an even integer. We have

$$B_n(s) \;=\; \frac{2}{D(s/2, K)}\int_0^\infty u^{-s-1}\; du \sum_{j=1}^{K}\binom{K}{j}(-1)^j b^n\left(u\sqrt{j}\right), \qquad (14)$$

$$\Delta_n(s) \;=\; \frac{2}{D(s/2, K)}\int_0^\infty u^{-s-1}\; du \sum_{j=1}^{K}\binom{K}{j}(-1)^j d^n\left(u\sqrt{j}\right). \qquad (15)$$

Remark: As for nonnegative even integers s not covered in the above theorem, the definitions (1, 2) immediately yield rational values for B_n, D_n respectively upon symbolic integration.

It is interesting that even though the goal of the above development is to provide practical quadrature formulae, we already have a byproduct:

Corollary 1 [Pole structure of the box integrals] $B_n(s)$, *for positive integer n, has precisely one pole, namely at $s = -n$. $\Delta_n(s)$, on the other hand, always has precisely $(n + 1)$ poles, located at $s = -2n, -2n + 1, \ldots, -n$.*

Proof: Clearly, by the original definitions (1) and (2), $B_n(s), \Delta_n(s)$ are both finite for nonnegative $\Re(s)$ so it suffices to look at negative $\Re(s)$ in any case. For $B_n(s)$, relation (11) has a pole factor at $s = -n$.

As for $\Delta_n(s)$, the solution (A_k) to the system of equations is unique since the matrix with entries $(-1)^k/(k+1)^{j-1}$ for $1 \le k, j \le n$ is nonsingular with determinant $(-1)^{\lfloor n/2 \rfloor} \prod_{i=1}^{n-1} \frac{i!}{(n+1)!}$ (see sequence A002109 in Sloane's *Online Encyclopedia of Integer Sequences*). Poles of Δ_n force the prefactor $\sum_{k=1}^{n+2} A_k k^{n+s}$ to vanish exactly when $n + s \in [-n, 0]$, thus giving a total of $(n+1)$ poles on the negative real s-axis. It remains to check that $\sum_{k=1}^{n+2} A_k k^{n+s}$ has no other zeros. This runs as follows. Consider the more general exponential sum

$$\Phi_N(t) := \sum_{j=1}^{N} \alpha_j \beta_j^t = 1$$

for arbitrary real constants α_j and positive β_j. If $\Phi_N(t) = 1$ has more than N solutions then Rolle's theorem and some manipulation will produce a derived system with at least N solutions and one less term. We are done, since when $N = 1$ this is impossible. **QED**

By employing the quadrature options embodied in Theorem 1, we have been able to calculate various $B_n(s), \Delta_n(s)$ values to extreme precision.[4]

One may also use the quadrature prescription to create contiguous plots of box integrals, as in Figure 1. Referring to Figure 1, note that both plots show the "natural" pole at $s = -2$, where the standard expectation integral diverges for $n = 2$ dimensions. However, we see that Δ_2 also has poles at $s = -3, -4$ as in Corollary 1.

3 Quadrature experiments

We employed the quadrature schema of Theorem 1 to effect extreme-precision values. The values shown to 50-digit accuracy in Section 7 were computed to over 400 digits. We might mention here that in fact this research began in earnest with the discovery, using the PSLQ integer relation algorithm [15],[8], that the extreme-precision numerical value we were able to compute for $\Delta_3(-1)$, namely

$$\Delta_3(-1) = 1.88231264438966016010560083886836758785246288031070\ldots \quad (16)$$

was (experimentally) given by the formula

$$\Delta_3(-1) \stackrel{?}{=} \frac{2}{5} + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} - \frac{2}{3}\pi - 6\log 2 + 2\log\left(1 + \sqrt{2}\right) \quad (17)$$

$$+ 12\log\left(1 + \sqrt{3}\right) - 4\log\left(2 + \sqrt{3}\right). \quad (18)$$

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[4]As in previous works, by "extreme precision" we mean, loosely speaking, enough precision to resolve a given entity into fundamental (or at least previously studied) constants; in the modern era this usually means precision to hundreds to thousands of decimal digits.

Soon we were able to prove this evaluation.[5] Upon further experimentation and analytic discoveries, we realized that not only do some B_n for various n have constants in common—such as log-surds, π, and so on—but the Δ_n constants also have many constants in common with themselves and the B_n. Such experimentation/detection moved us to find analytic relations between the various entities, as we describe in this paper.

Our numerical approach was to utilize the ARPREC arbitrary precision software package [10] to implement the four formulas of Theorem 1, namely (11), (13), (14) and (15), together with the definitions of the functions $b(u)$, $d(u)$ and $D(\rho, K)$ defined by formulas (4), (3) and (10). As stated in Theorem 1, these formulas are valid for all s except for positive even integers. For positive even arguments the integration in (1, 2) is trivial and the relevant box integral is rational, obtainable quickly with symbolic processing.[6]

Computation of $b(u)$ is entirely straightforward, once one has in hand an implementation of the erf function (see [2, pg. 297–298]).

Computation of $d(u)$ is complicated by the fact that for small arguments, cancellation of terms in the numerator typically results in severe relative numerical error. This can be ameliorated by employing the Taylor series for $d(u)$:

$$
\begin{aligned}
d(u) &= 1 - \frac{u^2}{6} + \frac{u^4}{30} - \frac{u^6}{168} + \frac{u^8}{1080} - \frac{u^{10}}{7920} + \frac{u^{12}}{65520} - \cdots \quad (19) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n+1) \cdot (n+1)!} \quad (20)
\end{aligned}
$$

Computation of the A_k coefficients was achieved by employing an arbitrary precision version of the well-known Linpack program for solving linear equations via LU decomposition [19].

The four formulas (11), (13), (14) and (15) of Theorem 1 themselves involve significant numerical difficulties, as they involve near-cancellation of terms for small arguments. Nonetheless we were able to compute the $B_n(s)$ and $D_n(s)$ to approximately 470-digit accuracy, utilizing 500-digit working precision, by using Gaussian quadrature, after splitting the integral into the intervals $(0, 1)$ and $(1, \infty)$, and applying the simple substitution $w = 1/u$ on the second integral.

We should add here that tanh-sinh quadrature [23], which we have used in numerous other studies of this sort, was not needed in this case, because each of the integrand functions in Theorem 1 are regular on the intervals of integration. In addition, the tanh-sinh scheme suffers here from the cancellation difficulties mentioned above, since it relies on evaluating the function very close to endpoints of the interval.

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[5]That is to say, the $\overset{?}{=}$ became $=$. In the present work, we use $=$ to mean rigorously proven, even though some of our more recondite expressions were first found in the $\overset{?}{=}$ sense, then later proven.

[6]*Mathematica* and *Maple* were both used quite heavily for much of this paper's development, and most of the work was symbolic. We have endeavored to check all results numerically, even after obtaining exact, symbolic forms. That being said, elsewhere in this paper we have for efficiency eschewed comments about which computer algebra system was used where.

4   Analytic continuation for the B_n

In our previous work [5] we demonstrated that the B_n expectations can always be reduced by at least one dimension; for example, via vector-field algebra, B_n can be obtained from an integral over the $(n-1)$-dimensional unit cube, like so for $n > 1$:

$$B_n(s) \;=\; \frac{n}{n+s} \int_{\vec{r}\in[0,1]^{n-1}} (1+r^2)^{s/2} \, \mathcal{D}\vec{r}. \tag{21}$$

Note that *already* with this reduction we see that B_n has a single, simple pole at $s = -n$, as the integral exists for all complex s.

With the advantage of hindsight, we define right now a general and quite useful function

$$C_{n,k}(s,a) \;:=\; \int_{\vec{r}\in[0,1]^n} r_1 r_2 \cdots r_k \, (a+r^2)^{s/2} \, \mathcal{D}\vec{r}, \tag{22}$$

and we also define $C_{0,0}(s,a) := a^{s/2}$. In this way we can begin developing relations for the B_n, as follows:

$$B_1(s) \;=\; \frac{1}{s+1} C_{0,0}(s,1) \;=\; \frac{1}{s+1}, \tag{23}$$

$$B_2(s) \;=\; \frac{2}{2+s} C_{1,0}(s,1) \;=\; \frac{2}{2+s}\, {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right), \tag{24}$$

$$B_3(s) \;=\; \frac{3}{3+s} C_{2,0}(s,1) = \frac{6}{(3+s)(2+s)} \int_0^{\pi/4} \left((1+\sec^2 t)^{s/2+1} - 1\right). \tag{25}$$

Note that the apparent pole at $s = -2$ for $B_3(s)$ is specious; the $1/(s+2)$ factor cancels in this case, to yield

$$B_3(-2) \;=\; 3 \int_0^{\pi/4} \log(1+\sec^2 t) \, dt. \tag{26}$$

It will be important later to observe a recurrence for the C-function in (25), which recurrence will also play a major role in our later development of the Δ_n box integrals.

For $n = 4$ and higher dimensions, such relations for the B_n rapidly become unwieldy. Take $n = 4$, and note that $C_3(s,1)$ is a 3-dimensional integral which yields, after integrating first in polar coordinates (r,ϕ) to handle two of the dimensions,

$$B_4(s) \;=\; \frac{4}{4+s} C_{3,0}(s,1) \tag{27}$$

$$=\; \frac{8}{4+s}\frac{1}{2+s} \int_0^{\pi/4} d\phi \int_0^1 \left((1+\sec^2\phi + z^2)^{s/2+1} - (1+z^2)^{s/2+1}\right) \, dz.$$

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Again the specious pole factor for $s = -2$ cancels, giving the special case

$$B_4(-2) \; = \; 2 \int_0^{\pi/4} d\phi \int_0^1 \left(\log(1 + \sec^2 \phi + z^2) - \log(1 + z^2) \right) \; dz. \qquad (28)$$

Though we are left with some stultifying integrals, the complete continuations above for B_1, B_2, B_3, B_4 do indicate some powerful experimental-mathematical pathways. Later we shall be able to build upon previous literature results by adding new closed forms to our knowledge.

There is another form of analytic relation also derived previously [5], namely

$$B_n(s) \; = \; \frac{n^s}{n+s} \sum (-s/2)_k \left(\frac{2}{n} \right)^k \beta_{n-1,k}, \qquad (29)$$

where the β coefficients arise from the b function, implicitly as

$$\left(b(u) e^{tu^2} \right)^n \; =: \; \sum_{k \geq 0} \beta_{n,k} 2^k u^{2k}. \qquad (30)$$

and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol.

The β numbers enjoy the relations

$$\beta_{n,k} \; = \; \sum_{k_1 + \ldots k_n = k} \frac{1}{(2k_1 + 1)!!} \cdots \frac{1}{(2k_n + 1)!!},$$

$$\beta_{n,k} \; = \; \sum_{j=0}^{k} \frac{1}{(2k+1)!!} \beta_{n-1,k-j},$$

$$(1 + 2k/n) \, \beta_{n,k} \; = \; \beta_{n,k-1} + \beta_{n-1,k}, \qquad (31)$$

with the recursions here ignited by $\beta_{0,k} := \delta_{0,k}$ and $\beta_{n,1} = n/3$. Here !! denotes the double factorial function: $m!! = m(m-2)(m-4)\cdots 3 \cdot 1$ if m is odd, and $m!! = m(m-2)(m-4)\cdots 4 \cdot 2$ if m is even.

The main point is, for $n > 1$ the sum (29) is *linearly absolutely convergent* for all complex s. (The summand decay factor is essentially $(1 - 1/n)^k$.) Accordingly, the series gives the analytic continuation over the entire s-plane. This series representation can be used computationally, either in place of the quadrature formulae, or as an extreme-precision check on same. The analytic series does have one advantage over the quadrature formalism: One need not break the plane by polarity in $\Re(s)$; the series (29) always converges (and there is the correct single-pole factor $1/(n+s)$).

5   Theory of hyperclosure and of the C-functions

We have seen that the box integrals B_n are expressible in terms of C-functions. It will turn out that there exist recurrence relations between C-functions; this leads eventually to relations between Δ_n and B_n values, and closed forms in many cases.

To quantify for our purposes what is a closed form, we first establish[7]

Definition 1 [Ring of hyperclosure] *We define a* hypergeometric evaluation *in a classical sense—to be a complex number $\sum_{n\geq 0} a_n z^n$ where z is algebraic, a_0 is rational, and generally $a_{n+1} = r(n)\, a_n$ where r is a fixed rational-polynomial function. We then contemplate the* ring of hyperclosure, *generated by all hypergeometric evaluations under ordinary operators $(\cdot, +)$. An element of this ring is said to be* hypergeometrically closed *(*hyperclosed *for short). We also deem ∞ to be hyperclosed (so that hyperclosure not be denied at poles).*

As examples, any rational (a sum with one term, say $r := 0$) is hyperclosed, and since

$$\sqrt{2} = \,_2F_1(-1/2, 1, 1; -1), \quad \pi = 4\,_2F_1(1/2, 1, 3/2, -1), \quad \log(1+z) = z\,_2F_1(1, 1, 2; -z),$$

it follows that combinations such as $\sqrt{2} + \pi$, $\pi\sqrt{2}$, $\pi^{10}\log^7 2$, $\frac{1}{3}\log(1+\sqrt{2})$ are all hyperclosed. Relevant to the present work is the fact of the Lerch transcendent $\Phi(z, s, a) := \sum_{n\geq 0} z^n/(n+a)^s$ being hyperclosed for algebraic z, integer s, and rational a. Thus, we shall have hyperclosure for any polylogarithm at algebraic argument, so perforce for the Lewin arctan integral Ti_2 and the Clausen function Cl_2 we discuss and employ later. Our box-integral tables have hyperclosed entries, except where "dangling integrals" remain when dimension $n > 4$. (Although we admit such unresolved integrals may still belong to the ring.)

An interesting sidelight looms here: What numbers are *not* hyperclosed? Certainly we need such numbers to exist, lest our entire research program of pointing out hyperclosure cases be a vacuous exercise. Given the algebraic character of parameter z in Definition 1, and the constraint that the succession ratio $r(n)$ be rational-polynomial, and the fact of all group generations under $(\cdot, +)$ yielding finite strings, we conclude that the ring of hyperclosure is countably infinite. Therefore: *Almost all complex numbers are not hyperclosed.* Equivalently, the ring of hyperclosure is a null set. As often happens in such analyses, we are stultified by the question: "If non-hyperclosed numbers are so overwhelmingly abundant, what is an example of such a number?" Well, we do not presently know a single such number. Perhaps π^π is not hyperclosed, but such questions loom Hilbertian in their evident profundity.

It is evident that the lowest-lying C-functions allow direct hypergeometric evaluation, so that $B_1(s)$, $B_2(s)$ are hyperclosed for any integer s, but that already for $C_{2,0}$ we run into some difficulty with the integration. Happily, there are some powerful connection formulae that we prove presently, starting with

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[7]It has occurred to the present authors that the age-old notion of "closed form" might be addressable, as a separate research program, vis such as our Definition 1. It is a pleasant task to be able to talk about closed forms with some semblance, at least, of rigor.

Theorem 2 [Convergent series for C-functions] *For all complex s, integer $n > 1$, and $\Re(a) > 0$ we have*

$$C_{n,0}(s,a) \;=\; (n+a)^{s/2} \sum_{k \geq 0} \left(\frac{2}{n+a} \right)^k (-s/2)_k \; \beta_{n,k} \;, \tag{32}$$

which expansion being linearly convergent with k-th summands being $O\left(((n-1)/(n+a))^k \right)$.

Proof: This expansion can be established using the same methods as for expansion (29), which is the case $a = 1$ as shown in [5]. **QED**

And now for the recurrence:

Theorem 3 [Fundamental relations for C-functions] *Special instances of the C-function are*

$$C_{n,0}(s,0) \;=\; B_n(s), \tag{33}$$
$$C_{0,0}(s,a) \;:=\; a^{s/2}, \tag{34}$$
$$C_{n,0}(s,1) \;=\; \frac{n+s+1}{n+1} B_{n+1}(s). \tag{35}$$

More generally, for all complex s, positive integer n, and positive real a we have

$$asC_{n,0}(s-2,a) \;=\; (s+n)C_{n,0}(s,a) - nC_{n-1,0}(s,a+1). \tag{36}$$

Another recurrence, involving now the second index on C, is, for positive integers a,k,

$$sC_{n,k}(s-2,a) \;=\; C_{n-1,k-1}(s,a+1) - C_{n-1,k-1}(s,a). \tag{37}$$

Proof: The recurrence (36) is proved by invoking β-relation (31) to show the right-hand side here in the theorem is $2a(\partial C_{n,0}(s,a)/\partial a)$ which equals the left-hand side. The recurrence (37) is easier, requiring simple integration by parts in the integral representation (22) for C.

QED

We are now prepared to establish a central result that will lead to a host of closed forms, which result being:

Theorem 4 *For any integer s and positive integer a, all the numbers*

$$C_{n,0}(s,a) \;:=\; \int_{\vec{r} \in [0,1]^n} (a+r^2)^{s/2} \; \mathcal{D}\vec{r} \tag{38}$$

for $n = 0,1,2$ are hyperclosed.

Remark: As with other parts of the present treatment, we not only prove hyperclosure, but do this constructively, so that an algorithm for achieving closed forms is intrinsic to the proof.

Proof: $C_{0,0}(s,a) := a^{s/2}$ and $C_{1,0}(s,a) = a^{s/2} {}_2F_1(1/2, -s/2; 3/2, -1/a)$ are clearly hyperclosed. So we turn our attention to $C_{2,0}$: For nonnegative even integer s, it is evident that the defining integral for $C_{2,0}$ yields rational values. Consider, then, a single evaluation for negative even s, where we employ polar coordinates in the defining 2-dimensional integral to obtain

$$C_{2,0}(-2, a) = \int_0^{\pi/4} dt \ \left(\log(a + \sec^2 t) - \log a\right) \tag{39}$$

$$= -G + \text{Ti}_2\left(\frac{2 + a - 2\sqrt{1+a}}{a}\right) + \frac{\pi}{2}\log\left(1 + \sqrt{1+a}\right) - \frac{\pi}{4}\log a,$$

where G is the Catalan constant and Ti_2 is the Lewin inverse-tangent integral, that is

$$\text{Ti}_2(x) := \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)^2} \tag{40}$$

$$= \frac{x}{4}\Phi(-x^2, 2, 1/2),$$

with Φ being the Lerch transcendent. (Note that $G = \text{Ti}_2(1)$.) Clearly $C_{2,0}(-2, a)$ is hyperclosed for positive algebraic a. Now, in recurrence (36) with $n = 2$, there is a term $C_{1,0}(s, a+1)$ which we have seen to be hyperclosed. The recurrence thus reveals that $C_{2,0}(-s, a)$ is hyperclosed for all negative even integers s.

It remains to handle all odd integers s. We observe another "ignition value"

$$\tag{41}$$
$$C_{2,0}(1, a) = -\frac{1}{3}a^{3/2}\tan^{-1}\left(\frac{1}{\sqrt{a^2 + 2a}}\right) + \frac{1}{6}(3a + 1)\log\left(\frac{a + 2\sqrt{a+2} + 3}{a+1}\right) + \frac{1}{3}\sqrt{a+2},$$

and this again drives the recurrence (36) to yield hyperclosure for any odd s. **QED**

6 Hyperclosure and the B_n

Theorem 5 [General interrelations for the B_n] *For integer $n > 1$ and all complex s, we have a recurrence*

$$(n + s)(n + s - 1)B_n(s) = s(n + s - 2)B_n(s - 2) + n(n - 1)C_{n-2,0}(s, 2), \tag{42}$$

and a higher-order variant

$$(n + s)(n + s - 1)(n + s - 3)B_n(s) = s(s - 2)(n + s - 4)B_n(s - 4) + \tag{43}$$
$$sn(n - 1)C_{n-2,0}(s - 2, 2) +$$
$$(n + s - 3)n(n - 1)C_{n-2,0}(s, 2).$$

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In addition, the residue Res_n of the unique pole at $B_n(-n)$ is given for all positive integers n by $\mathrm{Res}_1 = 1$ and for $n > 1$,

$$\mathrm{Res}_n \;\;=\;\; -2nB_n(-n-2) - n(n-1)C_{n-2,0}(-n,2). \tag{44}$$

Finally, we have a special analytic-continuation value valid for all positive integers n:

$$B_n(-n-1) \;\;=\;\; -nC_{n-2}(1-n,2). \tag{45}$$

Remark: In cases such as $s = -n$ where a pole is involved, an expression $(n+s)B_n(s)$, on the face of it $0 \cdot \infty$, is to be interpreted as the residue Res_n at the relevant pole. It turns out that the second recurrence allow us to traverse B_n-poles. Both the residue relation and the special analytic-continuation relation follow from the first recurrence.

Proof: The first B_n, C recurrence follows from the combination of proven relations (35) and (36). The second, higher-order recurrence is obtained by double application of the first recurrence. **QED**

Theorem 6 [Hyperclosure for specific B_n] *For any integer s, each of:*

$$B_1(s), B_2(s), B_3(s), B_4(s)$$

is hyperclosed. Moreover, the residues Res_n at the four poles $B_n(-n)$ for $n = 1, 2, 3, 4$ are also hyperclosed.

Remark: Again the proof following is constructive; i.e., gives rise to an immediate algorithm for generating closed forms.

Proof: From Theorem 4 it is immediate that $B_m(s) = \frac{m}{m+s}C_{m-1,0}(s,1)$ is hyperclosed for $m = 1, 2, 3$. So we turn to B_4, and observe that for nonnegative even integer s, the value $B_4(s)$ is rational. From Theorem 5 we have two recurrences

$$(3+s)(4+s)B_4(s) \;\;=\;\; s(s+2)B_4(s-2) + 12C_{2,0}(s,2), \tag{46}$$

$$(3+s)(4+s)(1+s)B_4(s) \;\;=\;\; s^2(s-2)B_n(s-4) + 12sC_{2,0}(s-2,2) + 12(s+1)C_{2,0}(s,2).$$

Inspection of these reveals that if we establish hyperclosure for $B_4(-1)$ and $B_4(-2)$, say, then we know all $B_4(\text{integer})$ to be hyperclosed via Theorem 4. (The first recurrence above is ignited by $B_4(-1)$, for all $B_4(\text{odd integer})$, while the second recurrence propagates even-integer arguments across the pole at $s = -4$.) To these ends, we observe

$$B_4(-1) \;\;=\;\; 2\log(3) - \frac{2}{3}G + 2\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) - \sqrt{8}\,\arctan\left(\frac{1}{\sqrt{8}}\right), \tag{47}$$

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and

$$B_4(-2) \;=\; \pi \, \log\left(2+\sqrt{3}\right) - 2\, G - \frac{\pi^2}{8}. \tag{48}$$

Starting with (27) and the evaluation of the jellium value J_4 given in our previous paper [5], the proof of (48) reduces to showing

$$\int_0^1 \frac{\log\left(3+s^2\right)}{1+s^2}\, ds = \frac{\pi}{4}\, \log\left(2+\sqrt{3}\right) + \frac{\pi}{4}\, \log\left(2\right) - \mathrm{Ti}_2\left(-2+\sqrt{3}\right) - G.$$

Using A2.2(3) in [18] this becomes

$$\int_0^1 \frac{\log\left(3+s^2\right)}{1+s^2}\, ds + \frac{\pi}{6}\, \log\left(2+\sqrt{3}\right) - \frac{\pi}{4}\, \log\left(2\right) - \frac{1}{3}\, G = 0.$$

Now substituting $s = \tan\theta$ yields an expression which computer algebra evaluates as

$$\frac{4}{3}\, G + \frac{\pi}{3}\, \log\left(2+\sqrt{3}\right) - \mathrm{Ti}_2\left(2+\sqrt{3}\right) - \mathrm{Ti}_2\left(2-\sqrt{3}\right) = 0.$$

This is easily confirmed using [18, A2.2(3),(4)]. The proof of (47) is similarly completed; said proof reduces to showing (63) noted below.

The hyperclosure of the Res_n now follows immediately from the residue recursion formula given in Theorem 5. **QED**

Another approach to recursions for B_n, Δ_n as a function of s (for all positive integer n) is via the linear differential equations satisfied by b and d see [12] and [11, p. 270]. This leads, for example, to an implementable 5-term recursion for $B_4(s)$ in terms of $B_4(s-2), B_4(s-4), B_4(s-6), B_4(s-8)$ with coefficients polynomial in s. (Note that tighter recursions are possible, as in Theorem 9 for $n = 5, 6$ dimensions, provided we only demand that terms be hyperclosed.)

7 Hyperclosure and the Δ_n

On the face of it, our definition (2) of the second box integral, repeated here:

$$\Delta_n(s) \;:=\; \int_{\vec{r},\vec{q}\in[0,1]^n} |\vec{r}-\vec{q}|^s \; \mathcal{D}\vec{r}\, \mathcal{D}\vec{q}$$

would appear to present a $(2n)$-dimensional integration problem. But this is not so; in fact we have quite generally

$$\Delta_n(s) = 2^n \int_{\vec{r}\in[0,1]^n} r^s \prod_{k=1}^n (1 - r_k) \; \mathcal{D}\vec{r}. \tag{49}$$

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This reduction from $2n$ to n dimensions can be derived in various ways. Perhaps the most intuitive is to look at the probability density of a coordinate difference $x = r_k - q_k$, which turns out to be a triangle distribution with base $[-1, 1]$ and height 1 at the origin-centered apex; i.e., the density is $1 - |x|$, and the new integral form follows.

It is interesting that (49) has "leading term" $2^n B_n(s)$ arising from the 1 component of the integrand's product, indicating heuristically that Δ_n and B_n should somehow be analytic relatives. Indeed, one can carry such an idea much further, to end up with powerful analytic expressions for $\Delta_n(s)$. Simple symmetries of the integrand in (49) reveal that

$$\Delta_n(s) = 2^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k C_{n,k}(s, 0). \tag{50}$$

On the other hand, we have the recursion (37) that may be used to decrement the both indices n, k on $C_{n,k}$. Thus we have the intriguing principle that Δ_n can be formally expressed in terms of $C_{m,0}$ functions, where $m \leq n$. But there is more:
Some thought in regard to relations (35, 36, 42) reveals that the relevant $C_{m,0}$ values lead back to the B_n box integrals, so that Δ_n can be formally expressed in terms of the B_n. The summary result is:

Theorem 7 [Δ_n as an analytic superposition of $B_2, \ldots, B_n$] *For arbitrary positive integer n and any complex s, we have an analytic superposition*

$$\Delta_n(s) = E_n(s) + \sum_{m=2}^{n} \sum_{j=1}^{m} R_{m,j}(s) B_m(s + 2n - 2j), \tag{51}$$

where E_n is an elementary function and each $R_{m,j}$ is a rational-polynomial function.

Remark: Thus Δ_n is in general a superposition of $(n+2)(n-1)/2$ different B terms.

Proof: This result follows from the combinatorics embodied in relations (50, 35, 36, 42). **QED**

Let us now give some entirely general analytic Δ_n evaluations; first the easy case $n = 1$, for which the sum in Theorem 7 is empty and we have a simple, elementary function:

$$\Delta_1(s) = 2 \int_0^1 (1 - x) x^2 \, dx = \frac{2}{1 + s} - \frac{2}{2 + s}. \tag{52}$$

Note the two poles, as we expect from previous analysis. Then from the combinatorics leading up to Theorem 7 we obtain

$$\Delta_2(s) = 8 \frac{(3 + s) 2^{\frac{s}{2} + 1} + 1}{(s + 2)(s + 3)(s + 4)} + 4 B_2(s) - \frac{4(s + 4)}{s + 2} B_2(s + 2). \tag{53}$$

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Note now the existence of three poles (at $s = -4, -3, -2$). Going yet deeper into the combinatorics, we have

$$\Delta_3(s) = \frac{24\left(-3^{\frac{s}{2}+2}s + 2^{\frac{s}{2}+3}s - 53^{\frac{s}{2}+2} + 52^{\frac{s}{2}+3} + 1\right)}{(s+2)(s+4)(s+5)(s+6)} + \frac{24}{s+2}B_2(s+2) -$$

$$\frac{24(s+6)}{(s+2)(s+4)}B_2(s+4) + 8B_3(s) - \frac{12(s+5)}{s+2}B_3(s+2) +$$

$$\frac{4(s+6)(s+7)}{(s+2)(s+4)}B_3(s+4).$$

There are some difficulties attendant on this Δ_3 form; namely, the apparent pole at $s = -2$ is specious. In order to get such as the $\Delta_3(-2)$ closed form in our tables, one needs to take a careful limit $s \to -2$. Note also that Δ_3 pole at $s = -3$ is caused exclusively by the $8B_3(s)$ term itself. Also, one needs the residue formula from Theorem 5 for the $B_3(s+4)$ term at $s = -7$.

At the next level we obtain the complete analytic continuation for Δ_4, in terms of B_2, B_3, B_4,:

$$\Delta_4(s) = \frac{64\left(\left(3 \cdot 2^{\frac{s}{2}+3} + 2^{s+6} - 3^{\frac{s}{2}+4}\right)s + 21 \cdot 2^{\frac{s}{2}+3} - 7 \cdot 3^{\frac{s}{2}+4} + 7 \cdot 2^{s+6} + 1\right)}{(s+2)(s+4)(s+6)(s+7)(s+8)} +$$

$$\frac{96}{(s+2)(s+4)}B_2(s+4) - \frac{96(s+8)}{(s+2)(s+4)(s+6)}B_2(s+6) + \frac{64}{s+2}B_3(s+2) -$$

$$\frac{96(s+7)}{(s+2)(s+4)}B_3(s+4) + \frac{32(s+8)(s+9)}{(s+2)(s+4)(s+6)}B_3(s+6) + 16B_4(s) -$$

$$\frac{88(s+6)}{3(s+2)}B_4(s+2) + \frac{8(s+8)(6s+43)}{3(s+2)(s+4)}B_4(s+4) -$$

$$\frac{8(s+8)(s+9)(s+10)}{3(s+2)(s+4)(s+6)}B_4(s+6). \tag{54}$$

Again the apparent pole at $s = -2$ is specious, so one must work out another logarithmic limit in that case. For $s = -9, -10$ we may use the known residues of the B-poles. (The poles at $s = -4, -5, -6, -7, -8$ *do* exist; e.g. the term with $B_3(s+2)$ has a pole at $s = -5$.)

Theorem 8 [Hyperclosure for specific Δ_n] *For any integer s, each of:*

$$\Delta_1(s), \Delta_2(s), \Delta_3(s), \Delta_4(s)$$

is hyperclosed.

Remark: Thus, with Theorem 6, we now know that $B_{1,2,3,4}$ and $\Delta_{1,2,3,4}$ are all hyperclosed at every integer argument. Our tables show exemplary closed forms.

Proof: From the specific instances of Theorem 7 displayed above for $n = 1, 2, 3, 4$, the hyperclosure of the relevant $B_n(\text{integer})$, together with special limits taken for the problematic cases $\Delta_3(-2)$, $\Delta_3(-7)$, $\Delta_4(-2)$, $\Delta_4(-9)$, $\Delta_4(-10)$ completes the proof. **QED**

8 Difficulties in $n \geq 5$ dimensions

We have not been able to fully obtain hyperclosure for $n = 5$ because we cannot completely handle the term $C_{3,0}$ that appears in the recursions at that level. That said, we can show hyperclosure for odd integers and report significant progress for the even cases. To begin with we can prove that

$$
B_5(1) = \frac{1}{6}\sqrt{5} + \frac{1}{360}\pi^2 - \frac{4}{3}\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) + \frac{10}{3}\log\left(\frac{\sqrt{5}+1}{2}\right)
$$
$$
+ \frac{7}{18}\left\{\mathcal{K}_1 - \frac{\pi}{2}\log\left(2 + \sqrt{3}\right)\right\},
$$

and

$$
B_5(-1) = \frac{5}{3}\left\{\mathcal{K}_1 - \frac{\pi}{2}\log\left(2 + \sqrt{3}\right)\right\} + \frac{1}{48}\pi^2 - 5\sqrt{6}\arctan\left(\frac{1}{\sqrt{15}}\right) + 5\log\left(\frac{\sqrt{5}+1}{2}\right).
$$

Here[8]

$$
\mathcal{K}_1 := \int_3^4 \frac{\text{arcsec}(x)}{\sqrt{x^2 - 4x + 3}}dx
$$

is the integral resolved in equation (65) below. This leads to the corresponding values in Table 4. More generally we have

$$
(2 + s)(4 + s)B_5(s) = 4\int_0^{\pi/4}\int_0^{\pi/4}\int_0^1 \left(\sec^2(a) + \sec^2(b) + z^2\right)^{2+s/2} dz\, da\, db - \tag{55}
$$
$$
2\pi\int_0^{\pi/4} {}_2F_1\left(\frac{1}{2}, -2 - \frac{s}{2}; \frac{3}{2}; -\cos^2(b)\right)\sec^{4+s}(b)\ db + \frac{1}{4}\frac{\pi^2}{5+s}.
$$

For integer s the integral in z can be performed symbolically and the hypergeometric function evaluated. This reduces $B_6(s)$ to a double integral expression for $s = -6$; and also for $s = -2, -4$ on applying l'Hôpital's rule.

---

[8]The numbering of this and subsequent $\mathcal{K}_m$ integrals is not ad hoc; rather, we use subscripts m consistent with the historically established $\mathcal{K}_{0,1,2,3}$ in what follows.

The values in Table 4 then can be pieced together from the tools provided above provided we admit the following still-to-be-resolved integrals:

$$\mathcal{K}_4 := \int_0^{\pi/4} \frac{\arctan\left(\frac{1}{\sqrt{2+\sec^2(t)}}\right)}{\sqrt{2+\sec^2(t)}} \, dt \tag{56}$$

and

$$\mathcal{K}_5 := \int_0^{\pi/4} \int_0^{\pi/4} \log\left(1+\sec^2(a)+\sec^2(b)\right) \, da \, db. \tag{57}$$

We note that (55) may be used to show that

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{\arctan\left(\frac{1}{\sqrt{\sec^2(a)+\sec^2(b)}}\right)}{\sqrt{\sec^2(a)+\sec^2(b)}} \, da \, db = 10 \, \mathcal{K}_4 - \frac{\pi}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{16} \pi^2,$$

and suggests that (57) should have a comparable one-dimensional form. Indeed

$$\begin{aligned}
\mathcal{K}_5 &= \int_0^1 \frac{\arctan\left(\sqrt{t}\right)\arctan\left(\frac{1}{\sqrt{3+t}}\right)}{\sqrt{3+t}\,(2+t)} dt - \frac{\pi}{2} \, G + \frac{\pi^2}{16} \ln(3) \tag{58}\\
&\quad + \frac{\pi}{2}\left(3-\sqrt{8}\right) \mathrm{Ti}_2\left(3-\sqrt{8}\right) + \frac{\pi}{12} \, \mathrm{Ti}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{8} \ln(\sqrt{2}-1).
\end{aligned}$$

Finally, as with lower dimensions, we can supply the following superposition formula for Δ_5 in terms of B_n with $2 \leq n \leq 5$,

$$\begin{aligned}
\Delta_5(s) &= 160 \, \frac{1+(9+s)\left(2^{6+s/2}+2^{10+s}-5^{4+s/2}-2\cdot 3^{5+s/2}\right)}{(2+s)\,(4+s)\,(6+s)\,(8+s)\,(9+s)\,(10+s)} \tag{59}\\
&\quad + \frac{320}{(2+s)\,(4+s)\,(6+s)} B_2\,(6+s) - \frac{320\,(10+s)}{(2+s)\,(4+s)\,(6+s)\,(8+s)} B_2\,(8+s)\\
&\quad + \frac{320}{(2+s)\,(4+s)} B_3\,(4+s) - \frac{480\,(9+s)}{(2+s)\,(4+s)\,(6+s)} B_3\,(6+s)\\
&\quad + \frac{160\,(10+s)\,(11+s)}{(2+s)\,(4+s)\,(6+s)\,(8+s)} B_3\,(8+s)\\
&\quad + \frac{160}{2+s} B_4\,(2+s) - \frac{880}{3}\frac{(8+s)}{(2+s)\,(4+s)} B_4\,(4+s) + \frac{80}{3}\frac{(10+s)\,(55+6\,s)}{(2+s)\,(4+s)\,(6+s)} B_4\,(6+s)\\
&\quad - \frac{80}{3}\frac{(10+s)\,(11+s)\,(12+s)}{(2+s)\,(4+s)\,(6+s)\,(8+s)} B_4\,(8+s)\\
&\quad + 32\,B_5\,(s) - 200\,\frac{(7+s)}{6+3\,s} B_5(2+s) + \frac{4}{3}\frac{(9+s)\,(291+35\,s)}{(2+s)\,(4+s)} B_5\,(4+s)\\
&\quad - \frac{8}{3}\frac{(10+s)\,(11+s)\,(47+5\,s)}{(2+s)\,(4+s)\,(6+s)} B_5\,(6+s) + \frac{4}{3}\frac{(10+s)\,(11+s)\,(12+s)\,(13+s)}{(2+s)\,(4+s)\,(6+s)\,(8+s)} B_5\,(8+s).
\end{aligned}$$

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Inspection of (59) shows that it establishes the hyperclosure of Δ_5(odd) on the basis of the hyperclosure of all B_5(odd). To summarize, and say something also about even integer arguments, we hereby state:

Theorem 9 [Partial hyperclosure for $n \geq 5$] *We have, for any integer s,*

$$2(s-2)s(s+1)\,B_5(s-4)-(s+3)\left\{s(3s+7)\,B_5(s-2)-(s+4)(s+5)\,B_5(s)\right\}=60C_{2,0}(s,3),$$

For $n = 6$, a similar superposition—this time involving four B-terms—is

$$-6s^2(s-2)(s-4)B_6(s-6)+(s-2)s(s+2)(18+11s)B_6(s-4)-$$

$$(3+s)(4+s)\left\{2s(8+3s)B_6(s-2)-(5+s)(6+s)B_6(s)\right\}=360\,C_{2,0}(s,4).$$

It follows that for any integer s, each of these B-superpositions is hyperclosed; moreover, for every odd integer s, each of $B_5(s)$ and $\Delta_5(s)$ is hyperclosed.

Proof: The two recurrences follow from a combination of (36, 42), and Theorem 4 implies the hyperclosure of the two superpositions. Since we have determined the pair $B_5(\pm 1)$ at the beginning of this section, the first recurrence in the theorem is thereby ignited, so $B_5(s)$ is hyperclosed for *all* odd s. Hyperclosure for Δ_5(odd) follows from (59). **QED**

Remark: Hence, we can assert that the only potentially problematic values for $n = 5$ are $B_5(-2n)$ and $\Delta_5(-2n)$ for positive integer n. In this setting both (55) and (59) engage a variety of thornier issues generated by the poles. But even here we discover that all B_5(even) can be resolved with the adjunction of $B_5(-6)$ (equivalently $\mathcal{K}_4$) and $B_5(-4)$ (equivalently $\mathcal{K}_5$), and this should yield in turn some, if not all, of the Δ_5(even). We should admit that the $n = 6$ recursion is basically all we know about 6 dimensional cases: We do not know a single nontrivial closed form for a $B_6(s)$ or $\Delta_6(s)$. We can, however, write

$$
\begin{aligned}
B_6\left(s\right) \;=\; & \frac{8}{(s+2)\,(s+4)\,(s+6)}\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\pi/4}\left(\sec^2\left(a\right)+\sec^2\left(b\right)+\sec^2\left(c\right)\right)^{3+s/2}\,da\,db\,dc \\
& -\frac{\pi}{2}\,\frac{3}{s+2}\,B_4\left(s+2\right)-\frac{\pi^2}{4}\,\frac{3}{(s+2)\,(s+4)}\,B_2\left(s+4\right),
\end{aligned}
$$

valid for all values of s. In particular

$$
\begin{aligned}
B_6\left(-7\right) =\; & -\frac{8}{15}\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\pi/4}\frac{da\,db\,dc}{\sqrt{\sec^2\left(a\right)+\sec^2\left(b\right)+\sec^2\left(c\right)}} \\
& +\frac{3\sqrt{2}}{5}\arctan\left(\frac{1}{\sqrt{8}}\right)\pi+\frac{3\sqrt{2}}{32}\pi^2.
\end{aligned}
$$

Moreover, the triple integral above resolves to

$$\frac{1}{2} \int_0^{\pi/4} \int_0^{\pi/4} \frac{\operatorname{arcsec}\left(1 + \sec^2(a) + \sec^2(b)\right)}{\sqrt{\sec^2(a) + \sec^2(b)}} \, da \, db = -\frac{3}{4} \int_1^2 \frac{\operatorname{arcsec}(z+3)\operatorname{arcsec}(z)}{(z+2)^{3/2}\sqrt{z+1}} \, dz$$
$$+ \frac{\pi}{\sqrt{8}} \left\{ 2 \arctan\left(\frac{3}{4}\right) - 3 \arctan\left(\frac{7}{8}\sqrt{2}\right) + \sqrt{\frac{3}{2}} \arctan\left(2\sqrt{6}\right) \right\}.$$

Thus we arrive at

$$B_6(-7) = \frac{2}{5}\mathcal{K}_8 + \frac{7\sqrt{2}}{60}\pi^2 - \frac{7\sqrt{2}}{5}\pi \arctan\left(\frac{1}{\sqrt{8}}\right) + \frac{8\sqrt{2}}{15}\pi \arctan\left(\frac{1}{2}\right) - \frac{2\sqrt{3}}{15}\pi \arctan\left(\sqrt{24}\right)$$

where

$$\mathcal{K}_8 = \int_1^2 \frac{\operatorname{arcsec}(x+3)\operatorname{arcsec}(x)}{\sqrt{(x+1)(x+2)^3}} \, dx. \tag{60}$$

Equivalently

$$B_6(-7) = -\frac{4}{3} \int_1^2 \frac{\operatorname{arcsec}(t)}{(t+2)(t+3)} \frac{\sqrt{t+1}}{\sqrt{t+4}} \, dt - \frac{5\sqrt{2}}{3}\pi \arctan\left(\frac{1}{\sqrt{8}}\right) + \frac{8\sqrt{2}}{9}\pi \arctan\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{36}\pi^2.$$

9 Curiosities and new definite integrals

One new result of some interest is that another unphysical, but valid analytically continued value discussed in our previous work [5], namely

$$B_4(-5) = -0.9612039326899534571216597800247 45\ldots$$

is now known in closed form. In fact, this previously mysterious analytic-continuation value is $-\sqrt{8}\arctan(1/\sqrt{8})$ (see Tables and Theorem 5). However, there are still many unexplained empirical phenomena. For example we do *not* know in closed form the zero of $\Delta_2(s)$ at

$$s = -5.1378771851212623537\ldots,$$

though we have developed herein enough machinery to find this zero to extreme precision. (See Figure 1 for a pictorial of the zero.) Moreover, we are not aware of any other zeros in $n = 2$ or any other dimension; evidently, this zero of Δ_2 is some kind of anomaly. There is also an empirical local minimum for B_2, at

$$s = 3.6675667756027541433\ldots,$$

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and again a closed form is unknown.

Another curiosity is that some statistical quantities over the hypercube are almost trivial to resolve. For example, the exponential expectation

$$E_n(\lambda) := \langle e^{-\lambda^2 |\vec{r}|^2} \rangle_{\vec{r} \in [0,1]^n}$$

for constant λ does *not* need to be obtained from a generating function or series. Instead, a closed form is immediate, based on the defining relation (3):

$$E_n(\lambda) = b^n(\lambda) = \left(\frac{\pi}{2\lambda}\right)^n \operatorname{erf}{}^n(\lambda).$$

The allure of this closed form is that it holds in all dimensions, and may well replace the box integrals $\langle |\vec{r} - \vec{q}|^s \rangle$ as the right tool to assess the statistical character of point clouds, again as in reference [14]. In spite of this streamlined approach to cloud measures, it seems (heuristically) that

$$\langle e^{-\kappa |\vec{r}|} \rangle_{\vec{r} \in [0,1]^n}$$

should be extremely difficult to evaluate in any general way. Even if one were to revert to series expansion of the exponential, one would need $B_n(s)$ for all nonnegative integers s, yet we have met with extreme difficulties when $n > 5$. This is not to claim some clever prescription—a novel use of recurrences, or differential relations—would not solve this problem.

It is highly interesting that the box-integral theory can sometimes be used to resolve previously unknown integrals that have, on the face of it, little to do with n-dimensional cubes. A canonical example is the case of $\Delta_3(-2)$, which is obtain through the rather delicate limit process $s \to -2$ in (54) in order to complete hyperclosure for $n = 3$. But when we attempted to evaluate $\Delta_3(-2)$ from the alternative formula

$$\Delta_3(-2) \;=\; 8 \int_{\vec{r} \in [0,1]^3} \frac{(1-x)(1-y)(1-z)}{x^2 + y^2 + z^2} \, dx \, dy \, dz, \tag{61}$$

we ended up with a single, troublesome arctan integral as the "dangling term." But the $s \to -2$ limit procedure does a complete hypergeometric breakdown, and all of this proves the peculiar result

$$\begin{aligned} \mathcal{K}_6 \;&:=\; \int_0^1 \sqrt{1+z^2} \arctan\left(\frac{1}{\sqrt{1+z^2}}\right) \, dz \tag{62} \\ &=\; \frac{1}{2} - \frac{G}{2} + \frac{\pi}{4} \log(1 + \sqrt{2}) + \frac{1}{2} \operatorname{Ti}_2(3 - 2\sqrt{2}). \end{aligned}$$

We do not have a direct proof of this integral relation, although the Lamb methods as discussed below should work in this case for a direct resolution of the integral. Another

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example of an indirect integral resolution is

$$\mathcal{K}_7 \; := \; \int_0^{\pi/4} \log(1 + \sqrt{2 + \sec^2 t}) \, dt$$

$$= \; -G + 2 \operatorname{Ti}_2 \left(3 - 2\sqrt{2}\right) + \frac{\pi}{2} \log\left(1 + \sqrt{2}\right). \tag{63}$$

In a set of marvelous preprints, G. Lamb [17] managed to largely resolve "dangling integrals" from our previous work [5]. We called these $\mathcal{K}_m$, for $m = 0, 1, 2, 3$, and now, via the combination of indirect proof (as with result (62)) and the Lamb theory, we know closed forms:

$$\mathcal{K}_0 := \int_0^1 \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{3+y^2}}\right)}{1 + y^2} \, dy = \frac{3}{2} \operatorname{Ti}_2 \left(3 - 2\sqrt{2}\right) + \frac{\pi}{4} \log\left(1 + \sqrt{2}\right) - \frac{1}{2}G. \tag{64}$$

The next integral is perhaps the most difficult one of the present treatment. Along the way to the following answer which started with Lamb's work several were stages of symbolics, at times involving over 28000 characters.

$$\mathcal{K}_1 \; := \; \int_3^4 \frac{\operatorname{arcsec}(x)}{\sqrt{x^2 - 4x + 3}} dx$$

$$= \; \operatorname{Cl}_2(\theta) - \operatorname{Cl}_2\left(\theta + \frac{\pi}{3}\right) - \operatorname{Cl}_2\left(\theta - \frac{\pi}{2}\right) + \operatorname{Cl}_2\left(\theta - \frac{\pi}{6}\right) - \operatorname{Cl}_2\left(3\theta + \frac{\pi}{3}\right) \tag{65}$$

$$+ \operatorname{Cl}_2\left(3\theta + \frac{2\pi}{3}\right) - \operatorname{Cl}_2\left(3\theta - \frac{5\pi}{6}\right) + \operatorname{Cl}_2\left(3\theta + \frac{5\pi}{6}\right) + \left(6\theta - \frac{5\pi}{2}\right) \log\left(2 - \sqrt{3}\right).$$

where $\operatorname{Cl}_2(\theta) := \sum_{n=1}^{\infty} \sin(n\theta)/n^2$ is the Clausen function, and

$$3\theta := \arctan\left(\frac{16 - 3\sqrt{15}}{11}\right) + \pi.$$

It may well be that the closed form in (65) can be further simplified.

Not so hard, but certainly nontrivial, is

$$\mathcal{K}_2 \; := \; \int_0^{\pi/4} \sqrt{1 + \sec^2(a)} \arctan\left(\frac{1}{\sqrt{1 + \sec^2(a)}}\right) da \tag{66}$$

$$= \; \frac{1}{2} \operatorname{Ti}_2\left(-2 + \sqrt{3}\right) + \frac{\pi}{8} \log\left(2 + \sqrt{3}\right) + \frac{\pi^2}{32}. \tag{67}$$

21

Finally, we have resolved a dangling integral from [5] involving dimension $N = 5$, as

$$\mathcal{K}_3 := \int_0^{\pi/4}\int_0^{\pi/4} \sqrt{1 + \sec^2(a) + \sec^2(b)}\, da\, db$$

$$= \frac{1}{4}\log(2+\sqrt{3}) + \frac{1}{48}\pi^2 - \frac{1}{10}B_5(-3). \tag{68}$$

Indeed the formula for the jellium constant J_5 given in [5, §7.3] yields precisely (68) and the closed form for $B_5(-3)$ is listed in Table 4.

Accordingly, the reader can see that our Tables have all dangling integrals removed save for $\mathcal{K}_4$ and $\mathcal{K}_5$ of (56) and (57).

And what about noninteger arguments for the box integrals? One might extend the definition of hypergeometric closure based on such findings as

$$\tag{69}$$

$$B_3\left(-\frac{7}{2}\right) = -2\pi - \frac{8}{5\cdot 2^{1/4}}F_1\left(\frac{5}{4};\frac{1}{2},\frac{3}{4};\frac{9}{4};\frac{1}{2},-\frac{1}{2}\right) + \frac{16}{5}\sqrt{\pi}\,\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}\,{}_2F_1\left(\frac{5}{4},\frac{3}{4};\frac{7}{4};-1\right).$$

Here, F_1 is the Appell hypergeometric function—defined as a series in two variables [2]. Thus we expect entities such as $B_3(\mathrm{odd}/2)$ to involve such as the Appell hypergeometric. Such examples suggest that there might be a kind of higher level of hypergeometric-closure theory for noninteger rational arguments s.

10 Open Questions

We conclude by reiterating the following open questions.

1. What is the simplest expression for $B_5(1)$, see (65)?

2. Can $\mathcal{K}_4$ and $\mathcal{K}_5$, given in (56), (57) be resolved as with (65)? Integer relation computations strongly suggest that these particular $\mathcal{K}$ integrals are rationally independent of each other and of the thirty or so other polylogarithmic constants used in our Tables. Still, one or both of $\mathcal{K}_4, \mathcal{K}_5$ could well be hyperclosed.

3. Is there a general evaluation (hyperclosed or not) of $B_n(-n-1)$ for all natural numbers n?

4. What transpires for noninteger but rational s in dimension $n \le 5$ (see relation (69)?

5. What transpires for dimension $n > 5$, say just for any of $B_6(1, -1, -2, -4)$?

6. Referring to Figure 1, what is a closed form for the mysterious zero? Are there any other zeros of any Δ_n whatsoever? Is there a closed form for the local minimum of $B_2(s)$ at $s \approx 3.667\ldots$?

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11 Selected numerical values and a compendium of proven closed forms

The following numerical values were calculated on the basis of the analytic formulae of expansion 29 (for $B_n(s)$ and Theorem 7 (for $\Delta_n(s)$). We separately verified the values below via the quadrature theorems.

The numerical reports below follow the row paradigm:

$n \quad s \quad B_n(s)$ or $\Delta_n(s)$

where the values are of course approximations to the implied precision. An entry "∞" signifies a pole.

To our knowledge, the closed forms displayed in our Tables exemplify the current state-of-knowledge about box integrals.[10] As we have intimated, a good portion of the analytic work was experimentally motivated, with computations involving either quadrature or our analytic series forms.

These numerical values are available to 460-digit accuracy in an online file, available at `http://crd.lbl.gov/~dhbailey/dhbpapers/box-int-b-delta.txt`.

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[10]Of course, other investigators may well have found new box-integral evaluations for $n \geq 4$; we are simply reporting on everything we presently know about such exact evaluations.

$$B_n(s):$$

$$
\begin{pmatrix}
1 & -8 & -0.142857142857142857142857142857142857142857142857 \\
1 & -7 & -0.166666666666666666666666666666666666666666666667 \\
1 & -6 & -0.200000000000000000000000000000000000000000000000 \\
1 & -5 & -0.250000000000000000000000000000000000000000000000 \\
1 & -4 & -0.333333333333333333333333333333333333333333333333 \\
1 & -3 & -0.500000000000000000000000000000000000000000000000 \\
1 & -2 & -1.000000000000000000000000000000000000000000000000 \\
1 & -1 & \infty \\
1 & 0 & 1.000000000000000000000000000000000000000000000000 \\
1 & 1 & 0.500000000000000000000000000000000000000000000000 \\
1 & 2 & 0.333333333333333333333333333333333333333333333333 \\
1 & 3 & 0.250000000000000000000000000000000000000000000000 \\
1 & 4 & 0.200000000000000000000000000000000000000000000000 \\
1 & 5 & 0.166666666666666666666666666666666666666666666667 \\
1 & 6 & 0.142857142857142857142857142857142857142857142857 \\
1 & 7 & 0.125000000000000000000000000000000000000000000000 \\
1 & 8 & 0.111111111111111111111111111111111111111111111111
\end{pmatrix}
$$

$$
\begin{pmatrix}
2 & -8 & -0.158201197576123087807186893661792609831523508 66428 \\
2 & -7 & -0.202703943940143623661575383803390057928319635 47069 \\
2 & -6 & -0.272262155637021558052936408591226697696742315 59570 \\
2 & -5 & -0.392837100659193069111580201169360577380464409 82693 \\
2 & -4 & -0.642699081698724154807830422909937860524646174 92188 \\
2 & -3 & -1.414213562373095048801688724209698078569671875 37694 \\
2 & -2 & \infty \\
2 & -1 & 1.7627471740390860504652186499595846180563206565 2327 \\
2 & 0 & 1.000000000000000000000000000000000000000000000000 \\
2 & 1 & 0.765195716464212691344766016396496795865944067 87952 \\
2 & 2 & 0.666666666666666666666666666666666666666666666666 \\
2 & 3 & 0.627180784883514720865482452203631738536092056 2117 \\
2 & 4 & 0.622222222222222222222222222222222222222222222222 \\
2 & 5 & 0.642695907644586390763108835694205708308953605 6059 \\
2 & 6 & 0.685714285714285714285714285714285714285714285 1428 \\
2 & 7 & 0.751659951007697971225268785227864961497619759 35417 \\
2 & 8 & 0.843174603174603174603174603174603174603174603 17460
\end{pmatrix}
$$

$$\begin{pmatrix}
3 & -8 & -0.14265709853439893660495343821908958888762502708148 \\
3 & -7 & -0.20363237466416374660521498702407835416190961442623 \\
3 & -6 & -0.31367283896888641587715269870072514211592383559001 \\
3 & -5 & -0.55047452239396231879312800552427063484023165891632 \\
3 & -4 & -1.30562962705065479621033295376348068215643441083205 \\
3 & -3 & \infty \\
3 & -2 & 1.91853105561093300588807925628197103962798387555418 \\
3 & -1 & 1.19003868198977675332190867514207694499118060735749 \\
3 & 0 & 1.00000000000000000000000000000000000000000000000000 \\
3 & 1 & 0.96059195645505295942510795139380636024097690754572 \\
3 & 2 & 1.00000000000000000000000000000000000000000000000000 \\
3 & 3 & 1.10139570939140687812203403510547952078327963488153 \\
3 & 4 & 1.26666666666666666666666666666666666666666666666666 \\
3 & 5 & 1.50871919447020333982278229537769388577287141449824 \\
3 & 6 & 1.85079365079365079365079365079365079365079365079365 \\
3 & 7 & 2.32881677842924686139772054356969696033746494832343 \\
3 & 8 & 2.99619047619047619047619047619047619047619047619047
\end{pmatrix}$$

$$\begin{pmatrix}
4 & -8 & -0.12563436349399430730524937404239130051014157311927 \\
4 & -7 & -0.20464509989277002063138774111523531325513276563725 \\
4 & -6 & -0.38093748929629845974354870407833432802478483424972 \\
4 & -5 & -0.96120393268995345712165978002474521286412992715686 \\
4 & -4 & \infty \\
4 & -3 & 1.61165174043060016204196553385288683776948507513204 \\
4 & -2 & 1.07171351557546047067080867784433752101727793208326 \\
4 & -1 & 0.96741202124116589866183643817815839013593700929996 \\
4 & 0 & 1.00000000000000000000000000000000000000000000000000 \\
4 & 1 & 1.12189961871586097735161517556754270920080795643954 \\
4 & 2 & 1.33333333333333333333333333333333333333333333333333 \\
4 & 3 & 1.65643241408719212813885397830393461099968596633396 \\
4 & 4 & 2.13333333333333333333333333333333333333333333333333 \\
4 & 5 & 2.83229584680208278742730991594029439544103954461450 \\
4 & 6 & 3.86031746031746031746031746031746031746031746031746 \\
4 & 7 & 5.38442386177588613770314868395705398265702874225811 \\
4 & 8 & 7.66645502645502645502645502645502645502645502645502
\end{pmatrix}$$

$$
\begin{pmatrix}
5 & -8 & -0.11647898000255184125565408496769427683539077763030 \\
5 & -7 & -0.22813171666759375219040630420029270618098726061057 \\
5 & -6 & -0.60682485151797019893358129603679159220054507179516 \\
5 & -5 & \infty \\
5 & -4 & 1.13629198898227049989471014695998822724624232794430 \\
5 & -3 & 0.80012321133838410594859790093424520754091697421787 \\
5 & -2 & 0.76560088060035042048313592041746790597916235131578 \\
5 & -1 & 0.83959702989032600319598511030307138077628236791140 \\
5 & 0 & 1.00000000000000000000000000000000000000000000000000 \\
5 & 1 & 1.26240663503625185427789711515933820822880865736373 \\
5 & 2 & 1.66666666666666666666666666666666666666666666666666 \\
5 & 3 & 2.28204086723432228895565004831505453383645598114093 \\
5 & 4 & 3.22222222222222222222222222222222222222222222222222 \\
5 & 5 & 4.67251999294153838072072788500736814808739049822241 \\
5 & 6 & 6.93650793650793650793650793650793650793650793650793 \\
5 & 7 & 10.51572713082173115025120837696166506202597099947055 \\
5 & 8 & 16.24656084656084656084656084656084656084656084465608
\end{pmatrix}
$$

$$\Delta_n(s):$$

$$
\begin{pmatrix}
1 & -8 & 0.04761904761904761904761904761904761904761904761904761904190 \\
1 & -7 & 0.06666666666666666666666666666666666666666666666666666666666 \\
1 & -6 & 0.10000000000000000000000000000000000000000000000000000000000 \\
1 & -5 & 0.16666666666666666666666666666666666666666666666666666666666 \\
1 & -4 & 0.33333333333333333333333333333333333333333333333333333333333 \\
1 & -3 & 1.00000000000000000000000000000000000000000000000000000000000 \\
1 & -2 & \infty \\
1 & -1 & \infty \\
1 & 0 & 1.00000000000000000000000000000000000000000000000000000000000 \\
1 & 1 & 0.33333333333333333333333333333333333333333333333333333333333 \\
1 & 2 & 0.16666666666666666666666666666666666666666666665666666666666 \\
1 & 3 & 0.10000000000000000000000000000000000000000000000000000000000 \\
1 & 4 & 0.06666666666666666666666666666666666666666666666666666666666 \\
1 & 5 & 0.04761904761904761904761904761904761904761904761904761904190 \\
1 & 6 & 0.03571428571428571428571428571428571428571428571428571428571428571142 \\
1 & 7 & 0.02777777777777777777777777777777777777777777777777777777777 \\
1 & 8 & 0.02222222222222222222222222222222222222222222222222222222222 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
2 & -8 & 0.06822762472765136912416181596100754531885473598098 \\
2 & -7 & 0.09294083664636187447493686254021835923781416672697 \\
2 & -6 & 0.11301620751602874407058187812163559692898975412761 \\
2 & -5 & -0.07625461122391551217627668959137948571584722188715 \\
2 & -4 & \infty \\
2 & -3 & \infty \\
2 & -2 & \infty \\
2 & -1 & 2.97320959824737870252818566763957179801974547921061 \\
2 & 0 & 1.00000000000000000000000000000000000000000000000000 \\
2 & 1 & 0.52140543316472067833098235660724397491403156777900 \\
2 & 2 & 0.33333333333333333333333333333333333333333333333333 \\
2 & 3 & 0.24071655184644294933101668636257383726321804178524 \\
2 & 4 & 0.18888888888888888888888888888888888888888888888888 \\
2 & 5 & 0.15763829510824937966070505604305418020025125166100 \\
2 & 6 & 0.13809523809523809523809523809523809523809523809523 \\
2 & 7 & 0.12589697435888104076625281035494259662276709561212 \\
2 & 8 & 0.11873015873015873015873015873015873015873015873015 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
3 & -8 & 0.0814430512274355485048180459643938728356304613721\underline{3} \\
3 & -7 & 0.1937653667355690812533534563306378552492833944137\underline{4} \\
3 & -6 & \infty \\
3 & -5 & \infty \\
3 & -4 & \infty \\
3 & -3 & \infty \\
3 & -2 & 5.633715158136109311035512522250318299375646506929\underline{3} \\
3 & -1 & 1.882312644389660160105600838868367587852462880310\underline{70} \\
3 & 0 & 1.000000000000000000000000000000000000000000000000 \\
3 & 1 & 0.6617071822671762351558311332484135817464001357909\underline{5} \\
3 & 2 & 0.500000000000000000000000000000000000000000000000 \\
3 & 3 & 0.4138673456114323148485170770656078352018274582436\underline{5} \\
3 & 4 & 0.366666666666666666666666666666666666666666666666 \\
3 & 5 & 0.342768303135877383217092595301379792346157857942\underline{7} \\
3 & 6 & 0.334920634920634920634920634920634920634920634920\underline{63} \\
3 & 7 & 0.339790634345494859834568392531120106272382014566\underline{83} \\
3 & 8 & 0.356190476190476190476190476190476190476190476190\underline{47}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & -10 & 0.0335348691554693970608865291062326163918032903184\underline{1} \\
4 & -9 & 0.0682520384529753203804939929154514910886125554005\underline{65} \\
4 & -8 & \infty \\
4 & -7 & \infty \\
4 & -6 & \infty \\
4 & -5 & \infty \\
4 & -4 & \infty \\
4 & -3 & 8.408092681270421476059509043171266751063334173878\underline{70} \\
4 & -2 & 2.769629185138387150108791178789441302922911151499\underline{73} \\
4 & -1 & 1.481432636521064749748769140727658302570952634154\underline{86} \\
4 & 0 & 1.000000000000000000000000000000000000000000000000 \\
4 & 1 & 0.7776656535862671153379340946178195099628827244171\underline{3} \\
4 & 2 & 0.666666666666666666666666666666666666666666666666 \\
4 & 3 & 0.6145870332986842671958636359886159445271230107259\underline{9} \\
4 & 4 & 0.600000000000000000000000000000000000000000000000 \\
4 & 5 & 0.6140178671073746657816734108574326404875536507585\underline{8} \\
4 & 6 & 0.653968253968253968253968253968253968253968253968\underline{25} \\
4 & 7 & 0.721079229991783935708181878362099007277502634704\underline{98} \\
4 & 8 & 0.819788359788359788359788359788359788359788359788\underline{35}
\end{pmatrix}
$$

$$\begin{pmatrix}
5 & -12 & 0.017338641399107525955692735795080076912107584951536 7 \\
5 & -11 & 0.027499242154161234696357635287129839547093559013824 9 \\
5 & -10 & \infty \\
5 & -9 & \infty \\
5 & -8 & \infty \\
5 & -7 & \infty \\
5 & -6 & \infty \\
5 & -5 & \infty \\
5 & -4 & 10.686183281588472967440247031997471912168858033886 36 \\
5 & -3 & 3.4563588332774467349499284025312579391461520950363 1 \\
5 & -2 & 1.8482235615230255916665224053020280306600694376049 78 \\
5 & -1 & 1.2616838961294571352098391741073546613335036639283 49 \\
5 & 0 & 1.0000000000000000000000000000000000000000000000000 00 \\
5 & 1 & 0.8785309152674802771789167963362247495118969910897 68 \\
5 & 2 & 0.8333333333333333333333333333333333333333333333333 33 \\
5 & 3 & 0.8396367219583214899512789873574779553881319505049 48 \\
5 & 4 & 0.8888888888888888888888888888888888888888888888888 88 \\
5 & 5 & 0.9812078544140851027142466259654148213561750793317 30 \\
5 & 6 & 1.1230158730158730158730158730158730158730158730158 73 \\
5 & 7 & 1.3269500937244000240549182980314478332588687594302 87 \\
5 & 8 & 1.6132275132275132275132275132275132275132275132275 13
\end{pmatrix}$$

n	s	$B_n(s)$
any	even ≥ 0	rational: $B_2(2) = 2/3$
1	-1	∞
1	any	$\frac{1}{s+1}$
2	-4	$-\frac{1}{4} - \frac{\pi}{8}$
2	-3	$-\sqrt{2}$
2	-2	∞
2	-1	$2\log(1 + \sqrt{2})$
2	1	$\frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1 + \sqrt{2})$
2	3	$\frac{7}{5}\sqrt{2} + \frac{3}{20}\log(1 + \sqrt{2})$
2	any	$\frac{2}{2+s}\,{}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right)$

Table 1: Some relatively easy evaluations for $B_n(s)$. Every B_k(integer) for $k = 1, 2$ can be given a closed form (see Theorem 6).

n	s	$B_n(s)$
3	-5	$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$
3	-4	$-\frac{3}{2}\sqrt{2}\arctan\frac{1}{\sqrt{2}}$
3	-3	∞
3	-2	$-3G + \frac{3}{2}\pi\log(1+\sqrt{2}) + 3\ \mathrm{Ti}_2(3 - 2\sqrt{2})$
3	-1	$-\frac{1}{4}\pi + \frac{3}{2}\log\left(2 + \sqrt{3}\right)$
3	1	$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2 + \sqrt{3}\right)$
3	3	$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi - \frac{7}{20}\log\left(2 + \sqrt{3}\right)$
3	any	Integral representation (25), relation (35)

Table 2: Example evaluations of $B_3(s)$. Every B_3(integer) can be given a closed form, being as we can express every B_3 in terms of a function $C_{2,0}$ which we have shown to be hyperclosed at relevant arguments.

n	s	$B_n(s)$
4	-5	$-\sqrt{8}\,\arctan\left(\frac{1}{\sqrt{8}}\right)$
4	-4	∞
4	-3	$4\,G - 12\,\mathrm{Ti}_2(3 - 2\sqrt{2})$
4	-2	$\pi\,\log\left(2 + \sqrt{3}\right) - 2\,G - \frac{\pi^2}{8}$
4	-1	$2\log 3 - \frac{2}{3}\,G + 2\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) - \sqrt{8}\,\arctan\left(\frac{1}{\sqrt{8}}\right)$
4	1	$\frac{2}{5} - \frac{G}{10} + \frac{3}{10}\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10}\,\arctan\left(\frac{1}{\sqrt{8}}\right)$
4	any	Integral representation (27), recursions (42)

Table 3: Example evaluations of $B_4(s)$. Every B_4(integer) can be given a closed form, on the basis of recursions and "ignition values" such as the pair $B_4(-1), B_4(-2)$.

n	s	$B_n(s)$
5	-8	$\frac{25}{36}\,\mathcal{K}_4 - \frac{25}{288}\,\pi\,\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{5}{72}\arctan\left(\frac{1}{2}\right)$
5	-6	$5\,\mathcal{K}_4 - \frac{5}{8}\,\pi\,\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$
5	-5	∞
5	-4	$-5\,\mathcal{K}_5 - \frac{5}{2}\,\pi\,G + \frac{5}{4}\,\pi^2\,\log\left(1+\sqrt{2}\right) + \frac{5}{2}\,\pi\,\mathrm{Ti}_2\left(3-2\,\sqrt{2}\right)$
5	-3	$-\frac{110}{9}\,G - 10\,\log\left(2-\sqrt{3}\right)\,\theta - \frac{1}{8}\,\pi^2 + 5\,\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right)$ $-10\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta + \frac{1}{3}\,\pi\right) + 10\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta - \frac{1}{6}\,\pi\right)$ $+\frac{10}{3}\,\mathrm{Cl}_2\left(\theta + \frac{1}{6}\,\pi\right) + \frac{20}{3}\,\mathrm{Cl}_2\left(\theta + \frac{4}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\theta + \frac{5}{3}\,\pi\right) - \frac{20}{3}\,\mathrm{Cl}_2\left(\theta + \frac{11}{6}\,\pi\right)$
5	-2	$\frac{8}{3}\,B_5(-6) - \frac{1}{3}\,B_5(-4) + \frac{5}{2}\,\pi\,\log(3) + 10\,\mathrm{Ti}_2\left(\frac{1}{3}\right) - 10\,G$
5	-1	$-\frac{110}{27}\,G + \frac{10}{3}\,\log\left(2-\sqrt{3}\right)\,\theta + \frac{1}{48}\,\pi^2 + 5\,\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right)$ $+\frac{10}{3}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta + \frac{1}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta - \frac{1}{6}\,\pi\right)$ $-\frac{10}{9}\,\mathrm{Cl}_2\left(\theta + \frac{1}{6}\,\pi\right) + \frac{20}{3}\,\mathrm{Cl}_2\left(\theta + \frac{4}{3}\,\pi\right) - \frac{10}{3}\,\mathrm{Cl}_2\left(\theta + \frac{5}{3}\,\pi\right) + \frac{20}{9}\,\mathrm{Cl}_2\left(\theta + \frac{11}{6}\,\pi\right)$
5	1	$-\frac{77}{81}\,G + \frac{7}{9}\,\log\left(2-\sqrt{3}\right)\,\theta + \frac{1}{360}\,\pi^2 + \frac{1}{6}\,\sqrt{5} + \frac{10}{3}\,\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{4}{3}\,\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) +$ $\frac{7}{9}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta + \frac{1}{3}\,\pi\right) - \frac{7}{9}\,\mathrm{Cl}_2\left(\frac{1}{3}\,\theta - \frac{1}{6}\,\pi\right)$ $-\frac{7}{27}\,\mathrm{Cl}_2\left(\theta + \frac{1}{6}\,\pi\right) - \frac{14}{27}\,\mathrm{Cl}_2\left(\theta + \frac{4}{3}\,\pi\right) + \frac{7}{27}\,\mathrm{Cl}_2\left(\theta + \frac{5}{3}\,\pi\right) + \frac{14}{27}\,\mathrm{Cl}_2\left(\theta + \frac{11}{6}\,\pi\right)$
5	any	(55), recursions (42)

Table 4: Example evaluations of $B_5(s)$. Here $\theta = \arctan\left(\frac{16-3\sqrt{15}}{11}\right)$. It is proven that every $B_5(\text{odd})$ is hyperclosed. Two unresolved ("dangling") integrals K_4, K_5 that are not yet known to be hyperclosed, but may well be, serve to generate all the $B_5(\text{even})$, as exemplified above with $B_5(-2, -4, -6, -8)$. Note that $B_5(-2)$ here can be derived quickly from the relevant recurrence in Theorem 9.

n	s	$\Delta_n(s)$
any	even ≥ 0	rational: $\Delta_2(2) = 1/3$
1	-1, -2	∞
1	any	$\frac{2}{s+1} - \frac{2}{s+2}$
2	-5	$\frac{4}{3} + \frac{8}{9}\sqrt{2}$
2	-2,-3,-4	∞
2	-1	$\frac{4}{3} - \frac{4}{3}\sqrt{2} + 4\log(1 + \sqrt{2})$
2	1	$\frac{2}{15} + \frac{1}{15}\sqrt{2} + \frac{1}{3}\log(1 + \sqrt{2})$
2	any	Formula (53)

Table 5: Some relatively easy evaluations for $\Delta_n(s)$. Every Δ_k(integer) for $k = 1, 2$ can be given a closed form (see Theorem 8). (Note: The closed form for $\Delta_2(-1)$ here has been repaired w.r.t. the previous work [5].)

n	s	$\Delta_n(s)$
3	-7	$\frac{4}{5} - \frac{16\sqrt{2}}{15} + \frac{2\sqrt{3}}{5} + \frac{\pi}{15}$
3	-3,-4,-5,-6	∞
3	-2	$2\pi - 12\,G + 12\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) + 6\pi\log\left(1 + \sqrt{2}\right) + 2\log 2 - \frac{5}{2}\log 3 - 8\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$
3	-1	$\frac{2}{5} - \frac{2}{3}\pi + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} + 2\log\left(1 + \sqrt{2}\right) + 12\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) - 4\log\left(2 + \sqrt{3}\right)$
3	1	$-\frac{118}{21} - \frac{2}{3}\,\pi + \frac{34}{21}\,\sqrt{2} - \frac{4}{7}\,\sqrt{3} + 2\,\log\left(1 + \sqrt{2}\right) + 8\,\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	3	$-\frac{1}{105} - \frac{2}{105}\,\pi + \frac{73}{840}\,\sqrt{2} + \frac{1}{35}\,\sqrt{3} + \frac{3}{56}\,\log\left(1 + \sqrt{2}\right) + \frac{13}{35}\,\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	any	Formula (54)

Table 6: Example evaluations for $\Delta_3(s)$. Every Δ_3(integer) can be given a closed form (see Theorem 8).

n	s	$\Delta_n(s)$
4	-10	$-\frac{8}{45}\,\pi\,\sqrt{3} - \frac{20}{9}\,\pi\,\log(2) + \frac{4}{9}\,\pi^2 + \frac{4}{3}\,\log(2) + \frac{1}{15}\,\log(3) + \frac{8}{3}\,\mathrm{Ti}_2\left(3 - 2\,\sqrt{2}\right)$
4	-9	$\frac{16}{5}\,\pi\,\sqrt{3} - \frac{32}{3}\,\pi\,\log(2) - \frac{2}{3}\,\pi^2 + \frac{16}{5}\,\pi + 8\,\sqrt{2}\,\arctan\left(2\,\sqrt{2}\right) - 24\,\log(2) + \frac{2}{5}\,\log(3)$ $+ 12\,\pi\,\log\left(\sqrt{2} - 1\right) - 64\,\mathrm{Ti}_2\left(3 - 2\,\sqrt{2}\right) + \frac{160}{3}\,G$
4	$-4, \ldots, -8$	∞
4	-3	$-\frac{128}{15} + \frac{1}{63}\,\pi - 8\,\log\left(1 + \sqrt{2}\right) - 32\,\log\left(1 + \sqrt{3}\right) + 16\,\log(2) + 20\,\log(3)$ $-\frac{8}{5}\,\sqrt{2} + \frac{32}{5}\,\sqrt{3} - 32\,\sqrt{2}\,\arctan\left(\frac{1}{\sqrt{8}}\right) - 96\,\mathrm{Ti}_2\left(3 - 2\,\sqrt{2}\right) + 32\,G$
4	-2	$-\frac{16}{15}\,\pi\,\sqrt{3} + \frac{16}{3}\,\pi\,\log\left(1 + \sqrt{3}\right) - \frac{8}{3}\,\pi\,\log(2) + 4\,\pi\,\log\left(\sqrt{2} + 1\right) - \frac{2}{3}\,\pi^2 + \frac{4}{5}\,\pi$ $+ \frac{8}{5}\,\sqrt{2}\,\arctan\left(2\,\sqrt{2}\right) + \frac{2}{5}\,\log(3) + 8\,\mathrm{Ti}_2\left(3 - 2\,\sqrt{2}\right) - \frac{40}{3}\,G$
4	-1	$\frac{704}{195} - \frac{8}{39}\,\pi - \frac{100}{13}\,\log(3) + \frac{120}{13}\,\log(2) - \frac{8}{65}\,\sqrt{2} + \frac{128}{65}\,\sqrt{3}$ $-\frac{140}{13}\,\log\left(1 + \sqrt{2}\right) - \frac{32}{13}\,\log\left(1 + \sqrt{3}\right) + \frac{160}{13}\,\sqrt{2}\,\arctan\left(\frac{1}{\sqrt{8}}\right) + \frac{48}{13}\,G$
4	1	$-\frac{23}{135} - \frac{16}{315}\,\pi - \frac{52}{105}\,\log(2) + \frac{197}{420}\,\log(3) + \frac{73}{630}\,\sqrt{2} + \frac{8}{105}\,\sqrt{3}$ $+\frac{1}{14}\,\log\left(1 + \sqrt{2}\right) + \frac{104}{105}\,\log\left(1 + \sqrt{3}\right) - \frac{68}{105}\,\sqrt{2}\,\arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{4}{15}\,G + \frac{4}{5}\,\mathrm{Ti}_2\left(3 - 2\,\sqrt{2}\right)$
4	any	Formula (54)

Table 7: Example evaluations for $\Delta_4(s)$. Every $\Delta_4(\text{integer})$ can be given a closed form (see Theorem 8) .

n	s	$\Delta_n(s)$
5	-12	$-\frac{2}{3}B_2(-6) + \frac{1}{3}B_2(-4) + 4B_3(-8) - 3B_3(-6) + \frac{1}{6}B_3(-4) - 16B_4(-10) +$ $\frac{44}{3}B_4(-8) - \frac{5}{3}B_4(-6) + 32B_5(-12) - \frac{100}{3}B_5(-10) + \frac{129}{20}B_5(-8) - \frac{13}{90}B_5(-6) +$ $\frac{\pi}{108}\sqrt{3} - \frac{7}{800}$
5	-5,...,-10	∞
5	-3	$-\frac{12304}{63} - \frac{512}{21}\sqrt{2} + \frac{576}{7}\sqrt{3} + \frac{800}{21}\sqrt{5} - \frac{320}{3}B_2(3) + \frac{448}{3}B_2(5)$ $-320\,B_3(1) + 960\,B_3(3) - \frac{1792}{3}B_3(5) - 160\,B_4(-1) + \frac{4400}{3}B_4(1) - \frac{20720}{9}B_4(3)$ $+896\,B_4(5) + 32\,B_5(-3) + \frac{800}{3}B_5(-1) - 1488\,B_5(1) + \frac{14336}{9}B_5(3) - 448\,B_5(5)$
5	-1	$\frac{16388}{189} + \frac{1024}{189}\sqrt{2} - \frac{192}{7}\sqrt{3} - \frac{4000}{189}\sqrt{5} + \frac{64}{3}B_2(5) - \frac{192}{7}B_2(7) + \frac{320}{3}B_3(3)$ $-256\,B_3(5) + \frac{960}{7}B_3(7) + 160\,B_4(1) - \frac{6160}{9}B_4(3) + 784\,B_4(5) - \frac{1760}{7}B_4(7)$ $+32\,B_5(-1) - 400\,B_5(1) + \frac{8192}{9}B_5(3) - 672\,B_5(5) + \frac{1056}{7}B_5(7)$
5	1	$-\frac{1279}{567}G - \frac{4}{189}\pi + \frac{4}{315}\pi^2 - \frac{449}{3465} + \frac{3239}{62370}\sqrt{2} + \frac{568}{3465}\sqrt{3} - \frac{380}{6237}\sqrt{5}$ $+\frac{295}{252}\log(3) + \frac{1}{54}\log\left(1+\sqrt{2}\right) + \frac{20}{63}\log\left(2+\sqrt{3}\right) + \frac{64}{189}\log\left(\frac{1+\sqrt{5}}{2}\right)$ $-\frac{73}{63}\sqrt{2}\arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{8}{21}\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) + \frac{104}{63}\log\left(2-\sqrt{3}\right)\theta$ $+\frac{5}{7}\mathrm{Ti}_2\left(3-2\sqrt{2}\right) + \frac{104}{63}\mathrm{Cl}_2\left(\frac{1}{3}\theta + \frac{1}{3}\pi\right) - \frac{104}{63}\mathrm{Cl}_2\left(\frac{1}{3}\theta - \frac{1}{6}\pi\right)$ $-\frac{104}{189}\mathrm{Cl}_2\left(\theta + \frac{1}{6}\pi\right) - \frac{208}{189}\mathrm{Cl}_2\left(\theta + \frac{4}{3}\pi\right) + \frac{104}{189}\mathrm{Cl}_2\left(\theta + \frac{5}{3}\pi\right) + \frac{208}{189}\mathrm{Cl}_2\left(\theta + \frac{11}{6}\pi\right)$
5	any odd	Theorem 9

Table 8: Example evaluations of $\Delta_5(s)$. It is proven that every $\Delta_5(\text{odd})$ is hyperclosed. Yet, the situation for $\Delta_5(\text{even})$ remains open (see caption to Table 4). $\Delta_5(\text{even})$ can be linked to B values, as with the evaluation of $\Delta_5(-12)$ here—with the $(12+s)B_4(8+s)$ term of relation (59) requiring a residue calculation at the B_4-pole.
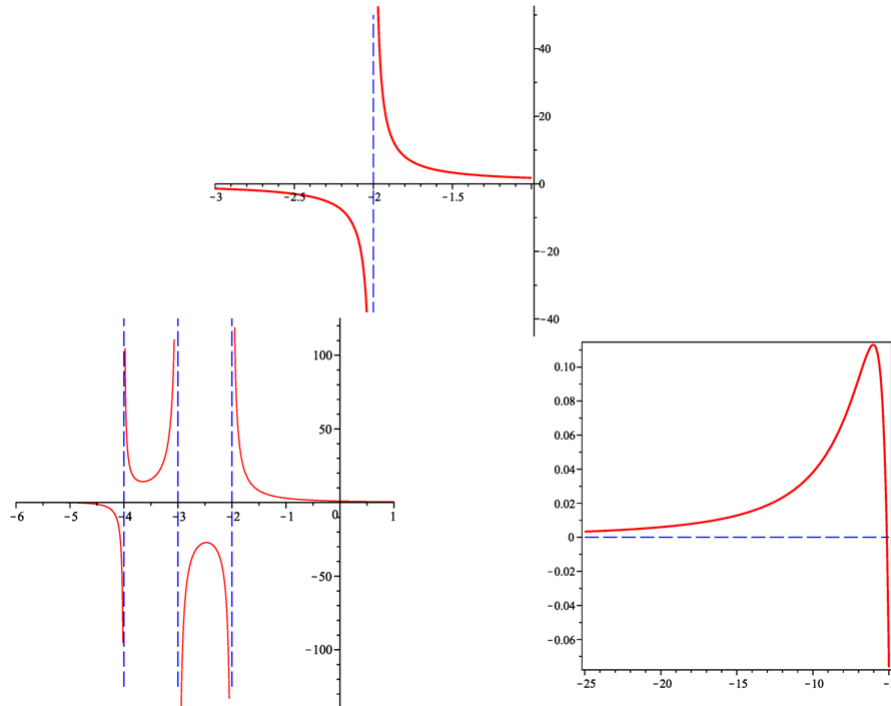
Figure 1: Analytic continuation: Plots of $B_2(s)$ (top) and $\Delta_2(s)$ (bottom) for real s in various intervals. $B_2(s)$ is the expected value of r^s where r is distance from origin to a random point on the unit square $[0,1]^2$. Likewise, $\Delta_2(s)$ is the expectation with r being the distance between two random points. $B_2(s)$ has a solitary pole at $s = -2$, while $\Delta(s)$ has three poles, at $s = -2, -3, -4$. Evidently, $\Delta_2(s)$ also has a mysterious zero at some real $s \approx -5.137\ldots$ (see text).