Relativistic Quantum Mechanics II

Lecture 35

Physics 342
Quantum Mechanics I

Friday, May 2nd, 2008

At the end of last time, we had the general Dirac equation for a charged particle in the presence of an electromagnetic field (below, we use $q\,\phi \longrightarrow \phi$ without assuming q is itself negative – this just gives us fewer constants to carry around):

$$\boxed{\begin{aligned} 0 &= \left[\left(i\,\frac{\partial}{\partial t} - \phi\right) - \mathbf{a}\cdot(-i\,\nabla - q\,\mathbf{A}) - \alpha\,m\right]\,\Psi(\mathbf{r},t) \\ \mathbb{I} &= a_j \cdot a_j = \alpha\,\alpha \\ 0 &= a_k\,a_\ell + a_\ell\,a_k \qquad k \neq \ell \\ 0 &= a_j\,\alpha + \alpha\,a_j. \end{aligned}} \tag{35.1}$$

All that's missing is a specification of the potential of interest, and operators $(\alpha, \mathbf{a})$ acting on the spin space of $\Psi(\mathbf{r},t)$. Once we have those, we can perform the usual temporal separation, and solve the eigenvalue problem to find the energy spectrum of the full relativistic form of Hydrogen.

35.1 Dirac Matrices

We had a set of (Pauli) spin matrices that acted on the spin state of the electron. Remember that for our non-relativistic Schrödinger equation, the spin of the electron was provided by tacking on a spinor, a combination of:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{35.2}$$

Then, while the Schrödinger equation did not directly involve the spin, we had the Pauli matrices that allowed us to operate on the spinor portion.

There, the spin and "orbital" wave functions were completely decoupled. In the relativistic Dirac setting, the "Hamiltonian" itself can potentially involve some analogue of the Pauli matrices. In fact, because of the expanded notion of "angular momentum" that exists in four-dimensional space-time, these end up being spinors with four components. The "Dirac" matrices serve to define the coefficients $(\alpha, \mathbf{a})$, and can be built from the Pauli matrices.

First let's review the Pauli matrix properties. These were constructed as a representation of angular momentum (meaning that the operators satisfy $[S_i, S_j] = i\,\epsilon_{kij} S_k$)

$$\sigma_x \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35.3}$$

As they stand, these are fine candidates for the components of $\mathbf{a}$, since:

$$\{\sigma_i\,,\,\sigma_j\} = 2\,\delta_{ij} \quad \sigma_i\,\sigma_i = \mathbb{I}. \tag{35.4}$$

But we do not have room, in this setting (two dimensional) to add in a fourth matrix α that satisfies the final anticommutation relation.

The Dirac matrices are defined via (the matrices themselves are four-dimensional, and we are displaying them in two-by-two sub-blocks):

$$\gamma^0 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i \doteq \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{35.5}$$

and these can be used to define:

$$\alpha = \gamma^0 \quad \mathbf{a} = \left(\gamma^0\right)^{-1} \boldsymbol{\gamma}. \tag{35.6}$$

These matrices satisfy the relations above, so that the Dirac equation, in 4×4 representation can be given an explicit form. This is not, strictly speaking, necessary, but makes working with the equation simpler.

35.2 Central Potential

Suppose we have the typical, time-independent Coulomb field $\phi(r) = -\frac{e^2}{r}$ as our electromagnetic contribution. Working in spherical coordinates, and separating out the time-dependence, Dirac's equation reads:

$$E\,\psi(\mathbf{r}) = \phi(r)\,\psi(\mathbf{r}) - i\,\mathbf{a}\cdot\nabla\psi(\mathbf{r}) + \alpha\,m\,\psi(\mathbf{r}). \tag{35.7}$$

Keep in mind that $\psi(\mathbf{r})$ has four components, and the $\mathbf{a}$ and α matrices are given above. There's not much we can do with the equation in its current form, so our first job is to find operators that commute with the Hamiltonian:

$$H_D = \phi(r) - i\,\mathbf{a}\cdot\mathbf{p} + \alpha\, m. \tag{35.8}$$

Since the potential is spherically symmetric, we should check $\mathbf{L}$, the angular momentum operator. Does this commute with H_D? If so, we will be able to find simultaneous eigenfunctions, and this would be useful since it implies that the wavefunction separates into an angular part and something else.

35.2.1 Angular Momentum

We know, from our work on the non-relativistic Hydrogen atom, that $[\mathbf{L}, V(r)] = 0$, and the final term in H_D clearly has $[\mathbf{L}, \alpha\, m] = 0$, since α does not talk to the "orbital" operators. So we are left with

$$[\mathbf{L}, H_D] = -i\,[\mathbf{L}, \mathbf{a}\cdot\mathbf{p}]. \tag{35.9}$$

In index notation, then, we have

$$[L_j, H_D] = -i\,[\epsilon_{jk\ell}\, r_k\, p_\ell, a_i\, p_i] = -i\, a_i\,[\epsilon_{jk\ell}\, r_k\, p_\ell, p_i]. \tag{35.10}$$

Evaluating the right-hand side is straightforward, keeping in mind that in our current units, $[r_i, p_j] = i\,\delta_{ij}$, we have

$$\begin{aligned} -i\, a_i\,[\epsilon_{jk\ell}\, r_k\, p_\ell, p_i] &= -i\, a_i\, \epsilon_{jk\ell}\,(r_k\, p_\ell\, p_i - p_i\, r_k\, p_\ell) \\ &= -i\, a_i\, \epsilon_{jk\ell}\,(i\,\delta_{ik}\, p_\ell + p_i\, r_k\, p_\ell - p_i\, r_k\, p_\ell) \\ &= a_i\, \epsilon_{ji\ell}\, p_\ell, \end{aligned} \tag{35.11}$$

so that

$$\boxed{[L_j, H_D] = \epsilon_{ji\ell}\, a_i\, p_\ell}. \tag{35.12}$$

That's not "good" in the sense that $\mathbf{L}$ does not commute with the Dirac Hamiltonian.

35.2.2 Spin

We know that the electron has spin, and that in the non-relativistic case, the spin operators (precisely the Pauli matrices) trivially commute with the

Hamiltonian, since Schrödinger's equation acts only on the "orbital" part of the wavefunction. So we should check the status of the commutator of H_D and $\mathbf{S}$ with:

$$\mathbf{S} = \frac{1}{2}\,\boldsymbol{\sigma}. \tag{35.13}$$

How can we make a representation of $\boldsymbol{\sigma}$ that operates in our four-dimensional space? It is easy to see that the four-by-four matrices:

$$S_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{35.14}$$

satisfy the spin commutation relations: $[S_i, S_j] = i\,\epsilon_{kij}\,S_k$ as matrices. Then the commutator with the Hamiltonian is

$$[S_j, H_D] = -i\,[S_j, a_i\,p_i] + m\,[S_j, \alpha] \tag{35.15}$$

where once again, the potential does not play a role in the commutation relation. Using the Dirac matrices, and our four-by-four S_j, we have: $[S_j, a_i] = i\,\epsilon_{jik}\,a_k$ and $[S_j, \alpha] = 0$. Then the commutator becomes:

$$[S_j, H_D] = -i\,p_i\,[S_j, a_i] = \;p_i\,\epsilon_{jik}\,a_k = -\epsilon_{ji\ell}\,a_i\,p_\ell, \tag{35.16}$$

precisely the opposite of (35.12) – that means that, while neither $\mathbf{L}$ nor $\mathbf{S}$ separately commutes, the sum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{35.17}$$

does:

$$[\mathbf{J}, H_D] = 0. \tag{35.18}$$

The stationary states of the Hamiltonian can be taken to be *total* angular momentum eigenstates. That's good, because we can now use J^2 and J_z as usual, with eigenvalues J and M, to simplify our analysis.

35.2.3 Parity

There is a final symmetry of the Hamiltonian we can exploit in forming our solution. The spatial reflection operator P takes: $P\,\psi(\mathbf{r}) = \psi(-\mathbf{r})$, i.e. $\mathbf{r} \longrightarrow -\mathbf{r}$. So for a time-indepedent Hamiltonian:

$$P\,H(\mathbf{p}, \mathbf{r}) = P\,H(i\,\hbar\,\nabla, \mathbf{r}) = H(-i\,\hbar\,\nabla, -\mathbf{r}) = H(-\mathbf{p}, -\mathbf{r}), \tag{35.19}$$

and if the Hamiltonian is invariant under this change, then $[P, H] = 0$. Let's see how this reads for our Dirac Hamiltonian – we'll put in a test function this time to keep everything straight:

$$
\begin{aligned}
\psi(\mathbf{r}) &= P\left(\phi(r) - i\,\mathbf{a}\cdot\mathbf{p} + \alpha\, m\right)\,\psi(\mathbf{r}) - \left(\phi(r) - i\,\mathbf{a}\cdot\mathbf{p} + \alpha\, m\right)\,\psi(-\mathbf{r}) \\
&= \left(\phi(-r) + i\,\mathbf{a}\cdot\mathbf{p} + \alpha\, m\right)\,\psi(-\mathbf{r}) - \left(\phi(r) - i\,\mathbf{a}\cdot\mathbf{p} + \alpha\, m\right)\,\psi(-\mathbf{r}) \\
&= 2\,i\,\mathbf{a}\cdot\mathbf{p}\,P\,\psi(\mathbf{r})
\end{aligned}
\tag{35.20}
$$

where we have used the spherical symmetry of the potential $\phi(r) = \phi(-r)$ (this is a strange notation, but reflects the fact that the distance from the origin for the vectors $\mathbf{r}$ and $-\mathbf{r}$ are identical).

So P itself does not commute with the Dirac Hamiltonian. But once again, we have to consider the "spin" operators together with the spatial ones. Note that:

$$
\begin{aligned}
[\alpha, H_D] &= -i\,[\alpha, \mathbf{a}\cdot\mathbf{p}] = -i\,p_j\,[\alpha, a_j] = -i\,p_j\,(\alpha\,a_j - a_j\,\alpha) \\
&= 2\,i\,p_j\,a_j\,\alpha \\
&= 2\,i\,\mathbf{a}\cdot\mathbf{p}\,\alpha
\end{aligned}
\tag{35.21}
$$

using the defining relation for $\alpha\,\mathbf{a}$ from (35.1).

Then if we define: $\hat{P} \equiv \alpha\,P$, we have

$$
\begin{aligned}
[\hat{P}, H_D] &= \alpha\,P\,H_D - H_D\,\alpha\,P = \alpha\,P\,H_D - \left(\alpha\,H_D - 2\,i\,\mathbf{a}\cdot\mathbf{p}\,\alpha\right)\,P \\
&= \alpha\,[P, H_D] + 2\,i\,\mathbf{a}\cdot\mathbf{p}\,\alpha \\
&= \alpha\,2\,i\,\mathbf{a}\cdot\mathbf{p} + 2\,i\,\mathbf{a}\cdot\mathbf{p}\,\alpha \\
&= 0,
\end{aligned}
\tag{35.22}
$$

and the operator $\hat{P}$ *does* commute with H_D. The eigenvalues of this operator are ± 1, as usual, since $\hat{P}\,\hat{P} = 1$, and we can take our wavefunction to be simultaneous eigenfunctions of J^2, J_z and $\hat{P}$.

35.3 Separation

Consider the wavefunction, now, split into two portions:

$$
\psi(\mathbf{r}) \doteq \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \doteq \begin{pmatrix} \Theta \\ \Phi \end{pmatrix},
\tag{35.23}
$$

then, the simultaneity of eigenfunctions means that $J^2\,\psi = J\,(J+1)\,\psi$, $J_z\,\psi = M\,\psi$, and $\hat{P}\,\psi = \pm 1\,\psi$. The advantage of the splitting above is that we can consider just the spatial parity P:

$$\hat{P}\,\psi = P \begin{pmatrix} \Theta \\ -\Phi \end{pmatrix} = \pm \begin{pmatrix} \Theta \\ -\Phi \end{pmatrix}, \tag{35.24}$$

and now we see that the two different components of the wave-function have opposite parities (opposite behavior under purely spatial reflection, in this case).

We know that the entire wavefunction must be in an eigenstate of J^2 – that means we should be constructing eigenspinors of the usual $\mathcal{Y}_J^M$ type, and we know that the parity of $\mathcal{Y}_J^{M+}$ and $\mathcal{Y}_J^{M-}$ are opposite one another for a given J, then we can consider a solution of the form:

$$\psi(\mathbf{r}) \doteq \begin{pmatrix} F(r)\,\mathcal{Y}_J^{M-} \\ -i\,f(r)\,\mathcal{Y}_J^{M+} \end{pmatrix}, \tag{35.25}$$

where we now have, clearly, the angular solutions, and parities in place. The radial functions are our eventual targets (our choice of $-i$ in the second term is conventional, and will make both radial equations real). Remember here that $\mathbf{S}$ just acts on each sub-spinor separately, so an object like J^2 is two copies of the usual J^2 for spin one half we studied earlier, one applied to Θ, one to Φ.

Now, finally, we must look at the full operator form – if we think about the way in which the Pauli matrices are embedded in the Dirac matrices, it is pretty clear that we can split the Dirac equation in two:

$$\begin{aligned} H_D\,\Theta &= (\phi(r) + m)\ \Theta - i\,\boldsymbol{\sigma}\cdot\nabla\,\Phi \\ H_D\,\Phi &= (\phi(r) - m)\ \Phi - i\,\boldsymbol{\sigma}\cdot\nabla\,\Theta. \end{aligned} \tag{35.26}$$

We already know to associate $\Theta \sim \mathcal{Y}_J^{M-}$ and $\Phi \sim \mathcal{Y}_J^{M+}$, and now it is clear that we have already solved the angular portion of the problem.

What we need now is a way to evaluate the $\boldsymbol{\sigma}\cdot\nabla$ terms – the angular portion of these act on the $\mathcal{Y}_J^{M\pm}$, and the radial part will define our ODE. We can also exploit the following identity for the Pauli matrices:

$$(\boldsymbol{\sigma}\cdot\mathbf{A})\,(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\,\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B}), \tag{35.27}$$

to write:

$$(\boldsymbol{\sigma}\cdot\mathbf{r})\,(\boldsymbol{\sigma}\cdot\mathbf{p}) = \mathbf{r}\cdot\mathbf{p} + i\,\boldsymbol{\sigma}\cdot\mathbf{L}. \tag{35.28}$$

In order to get $(\boldsymbol{\sigma} \cdot \mathbf{p})$ (reverting to $\mathbf{p}$ notation for ∇ to make things a little easier) by itself, we can use (35.27) on $(\boldsymbol{\sigma} \cdot \mathbf{r})(\boldsymbol{\sigma} \cdot \mathbf{r}) = r^2$, so we have

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{p}) &= \frac{1}{r^2}\,(\boldsymbol{\sigma} \cdot \mathbf{r})\,(\mathbf{r} \cdot \mathbf{p} + i\,\boldsymbol{\sigma} \cdot \mathbf{L}) \\ &= (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \left(\hat{\mathbf{r}} \cdot \mathbf{p} + \frac{i}{r}\,\boldsymbol{\sigma} \cdot \mathbf{L} \right). \end{aligned} \tag{35.29}$$

Finally, the term $\hat{\mathbf{r}} \cdot \mathbf{p} = \frac{\hbar}{i}\frac{\partial}{\partial r}$ as usual. We know, basically from our construction of $\mathcal{Y}_J^M$, that:

$$\boxed{\begin{aligned} \mathbf{S} \cdot \mathbf{L}\,\mathcal{Y}_J^{M-} &= \left(\frac{\hbar}{2}\right)\left(J - \frac{1}{2}\right)\mathcal{Y}_J^{M-} \\ \mathbf{S} \cdot \mathbf{L}\,\mathcal{Y}_J^{M+} &= \left(\frac{\hbar}{2}\right)\left(-J - \frac{3}{2}\right)\mathcal{Y}_J^{M+}, \end{aligned}} \tag{35.30}$$

and to get $\boldsymbol{\sigma} \cdot \mathbf{L}$, we just drop the $\frac{1}{2}\,\hbar$. The operator $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$ acts on the $\mathcal{Y}_J^{M\pm}$ via[1]:

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\,\mathcal{Y}_J^{M\pm} = -\mathcal{Y}_J^{M\mp}. \tag{35.33}$$

Using all of this, we can write down the radial equation for the Coulomb field:

$$\begin{aligned} 0 &= (E - m - \phi)\;F - \left(\frac{d}{dr} + \frac{J + \frac{3}{2}}{r}\right) f \\ 0 &= (E + m - \phi)\;f + \left(\frac{d}{dr} - \frac{J - \frac{1}{2}}{r}\right) F \end{aligned} \tag{35.34}$$

This is general, for any potential $\phi(r)$ – but we have the specific attractive

[1] This statement can be proved in a relatively straightforward manner – if you write

$$\boldsymbol{\sigma} \cdot \mathbf{r} = \frac{1}{2}\,(\sigma_+ + \sigma_-)\,\sin\theta\,\cos\phi + \frac{1}{2\,i}\,(\sigma_+ - \sigma_-)\,\sin\theta\,\sin\phi + \sigma_z\,\cos\theta, \tag{35.31}$$

and apply the spin operators to the $\chi_\pm$ appearing in $\mathcal{Y}_J^{M\pm}$, you get expressions of the form:

$$\sin\theta\,e^{\pm i\,\phi}\,Y_{J\pm\frac{1}{2}}^{M\pm\frac{1}{2}} \qquad \cos\theta\,Y_{J\pm\frac{1}{2}}^{M\pm\frac{1}{2}}. \tag{35.32}$$

But $\sin\theta\,e^{\pm i\,\phi}$ is $Y_1^{\pm 1}$ and $\cos\theta$ is Y_1^0, so these products can be represented in terms of angular momentum addition of spin $J \pm \frac{1}{2}$ and spin 1.

Coulomb case: $\phi(r) = -\frac{e^2}{r}$, so we have, setting $\lambda \equiv J + \frac{1}{2}$:

$$\begin{aligned} 0 &= \left(E - m + \frac{e^2}{r}\right) F - \left(\frac{d}{dr} + \frac{\lambda + 1}{r}\right) f \\ 0 &= \left(E + m + \frac{e^2}{r}\right) f + \left(\frac{d}{dr} - \frac{\lambda - 1}{r}\right) F. \end{aligned} \tag{35.35}$$

In our current units, where $\hbar = c = 1$, we measure mass in units of energy, the real term that appears above would be $m\,c^2$, the rest energy of the electron.

Going to the large r limit, where the potential and centrifugal terms are ~ 0, we have the simplified equations

$$(E - m)\,F - f' = 0 \qquad (E + M)\,f + F' = 0 \longrightarrow (E^2 - m^2)\,f + f'' = 0. \tag{35.36}$$

The asymptotic solution is then

$$f(r) \sim e^{\pm i\sqrt{E^2 - m^2}\,r} \qquad F(r) \sim \pm \frac{i\,e^{i\sqrt{E^2 - m^2}\,r}\,(E + m)}{\sqrt{E^2 - m^2}}. \tag{35.37}$$

To get a bound state, we must have both radial functions decaying at spatial infinity, and this suggests that $E^2 < m^2$ – that's perfectly reasonable, a bound state should have energy less than the rest energy of the electron. Then both radial equations have the form

$$f(r) \sim F(r) \sim e^{-\sqrt{m^2 - E^2}\,r} \tag{35.38}$$

at spatial infinity.

With the limit in hand, we make a series ansatz:

$$\begin{aligned} F(r) &= e^{-\sqrt{m^2 - E^2}\,r}\,r^\rho \sum_{j=0}^\infty \alpha_j\,r^j \\ f(r) &= e^{-\sqrt{m^2 - E^2}\,r}\,r^\rho \sum_{j=0}^\infty \beta_j\,r^j, \end{aligned} \tag{35.39}$$

and the derivatives are:

$$
\begin{aligned}
F'(r) &= e^{-\sqrt{m^2-E^2}\,r}\,r^\rho \left[-\sqrt{m^2-E^2}\sum_{j=0}^\infty \alpha_j\,r^j + \rho\sum_{j=0}^\infty \alpha_j\,r^{j-1} + \sum_{j=0}^\infty \alpha_j\,j\,r^{j-1}\right] \\
&= e^{-\sqrt{m^2-E^2}\,r}\,r^\rho \left[-\sqrt{m^2-E^2}\sum_{j=0}^\infty \alpha_j\,r^j + \sum_{k=-1}^\infty (\rho+k+1)\,\alpha_{k+1}\,r^k\right] \\
&= e^{-\sqrt{m^2-E^2}\,r}\,r^\rho \left[\frac{\rho\,\alpha_0}{r} + \sum_{j=0}^\infty\Big(-\sqrt{m^2-E^2}\,\alpha_j + (\rho+j+1)\,\alpha_{j+1}\Big)\,r^j\right]
\end{aligned}
\tag{35.40}
$$

and similarly for $f'(r)$. Inserting these into the first equation in (35.35), we get:

$$
\begin{aligned}
0 &= r^\rho\sum_{j=0}^\infty (E-m)\,\alpha_j\,r^j - r^\rho\left[\frac{\rho\,\beta_0}{r} + \sum_{j=0}^\infty\Big(-\sqrt{m^2-E^2}\,\beta_j + (\rho+j+1)\,\beta_{j+1}\Big)\,r^j\right] \\
&\quad + r^\rho\sum_{j=0}^\infty\big(\alpha_j\,e^2 - \beta_j\,(\lambda+1)\big)\,r^{j-1} \\
&= \sum_{j=0}^\infty\Big[(E-m)\,\alpha_j + \sqrt{m^2-E^2}\,\beta_j + e^2\,\alpha_{j+1} - (\lambda+1+\rho+j+1)\,\beta_{j+1}\Big]\,r^j \\
&\quad + \frac{1}{r}\big(e^2\,\alpha_0 - (\lambda+1)\,\beta_0 - \rho\,\beta_0\big)\,.
\end{aligned}
\tag{35.41}
$$

For the second:

$$
\begin{aligned}
0 &= \sum_{j=0}^\infty\Big[(E+m)\,\beta_j - \sqrt{m^2-E^2}\,\alpha_j + e^2\,\beta_{j+1} + (\rho+j+1-(\lambda-1))\,\alpha_{j+1}\Big]\,r^j \\
&\quad + \frac{1}{r}\big(e^2\,\beta_0 - (\lambda-1)\,\alpha_0 + \rho\,\alpha_0\big)\,.
\end{aligned}
\tag{35.42}
$$

Taking the $\frac{1}{r}$ part of each of these equations, we can solve for α_0 in terms of β_0, and then we find that

$$
\rho = -1 \pm \sqrt{\lambda^2 - e^4}. \tag{35.43}
$$

The negative root above leads to a singularity at the origin (remember that we must have $F^2\,r^2$ integrable at 0). So we find $\rho = -1 + \sqrt{\lambda^2 - e^4}$. In order to make sense of this equation, keep in mind that e^2/ℓ is an energy (in Gaussian units), so e^2 has units of energy$\times$ length. Now in order to get the unitless e^2 appearing above, we need to multiply by something that has units of 1/(energy$\times$length) – precisely $\hbar\, c$ in this case, since each of those is 1. The actual numerical value in the above equation can be recovered by taking $e = 4.8 \times 10^{(-10)}$ esu and forming $e^2/(\hbar\, c) \sim \frac{1}{137}$. The point is, for $\lambda = J + \frac{1}{2}$, even for $J = 0$, we have a positive number inside the square root.

Moving along to the recursion relation itself, we can solve for a_{j+1} in terms of b_{j+1} – that gives:

$$\begin{aligned}&\left(e^2\,\sqrt{m^2-E^2}+(E+m)\,(2+j+\lambda+\rho)\right)\,b_{j+1}\\ &=\left(e^2\,(m+E)-\sqrt{m^2-E^2}\,(2+j-\lambda+\rho)\right)\,a_{j+1}.\end{aligned} \tag{35.44}$$

By inputting this relation back into the recursion, it is possible to find a_{j+1} entirely in terms of a_j – that analysis suggests that this series must truncate. Suppose it does, if we have $a_{N+2} = b_{N+2} = 0$ for some N, then we know, again from the individual recursion relations, that

$$-\sqrt{m^2-E^2}\,a_{N+1}+(m+E)\,b_{N+1}=0 \qquad (E-m)\,a_{N+1}+\sqrt{-E^2+m^2}\,b_{N+1}=0$$
$$\longrightarrow \frac{\beta_{N+1}}{\alpha_{N+1}}=\sqrt{\frac{m-E}{m+E}} \tag{35.45}$$

and, in addition, we can set $j = N$ in (35.44) to get the relevant ratio there – then we have (using $\rho = -1 + \sqrt{\lambda^2 - e^4}$)

$$\frac{e^2\,\sqrt{m+E}-\sqrt{m-E}\,\left(1-\lambda+\sqrt{\lambda^2-e^4}+N\right)}{e^2\,\sqrt{m-E}+\sqrt{m+E}\,\left(1+\lambda+\sqrt{\lambda^2-e^4}+N\right)}=\sqrt{\frac{m-E}{m+E}}, \tag{35.46}$$

which can be simplified by cross-multiplication. Set $n = N + 1$ to denote the final non-zero term in the expansion (the principle quantum number),

$$e^2\,E=\sqrt{m^2-E^2}\,\left(n+\sqrt{\lambda^2-e^4}\right)\longrightarrow \frac{E}{m}=\left[1+\frac{e^4}{(n+\sqrt{\lambda^2-e^4})^2}\right]^{-1/2}. \tag{35.47}$$

With units, we have:

$$\boxed{\frac{E}{m\,c^2}=\left[1+\frac{\alpha^2}{\left(n+\sqrt{(J+\frac{1}{2})^2-\alpha^2}\right)^2}\right]^{-1/2}} \tag{35.48}$$

with $\frac{1}{\alpha} \sim 137$, the fine structure constant.

In order to compare with the Coulomb case, we have to subtract off the rest energy of the electron, which is built into this expression. In addition, the n appearing above is no the same n that labels the non-relativistic Hydrogen wavefunction. If we expand in small α:

$$(E - m\,c^2) \sim -\frac{m\,c^2\,\alpha^2}{\left(n + \left(J + \frac{1}{2}\right)\right)^2} = -\frac{m\,e^4}{32\,\pi^2\,\epsilon_0^2\,\hbar^2\left(n + J + \frac{1}{2}\right)^2}, \tag{35.49}$$

and we see that the Coulomb n is $n_C = n + J + \frac{1}{2}$. Then we can write the final relativistic form for comparison:

$$\boxed{E = m\,c^2\left[1 + \frac{\alpha^2}{\left(n_C - \left(J + \frac{1}{2}\right) + \sqrt{(J + \frac{1}{2})^2 - \alpha^2}\right)^2}\right]^{-1/2} - m\,c^2.} \tag{35.50}$$