

**Problem 2.1**

(10 points, 3.3 per part)

- a. Find the Fourier transform of the function:  $f(x) = e^{-|x|}$ .

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|x|} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^0 e^{x(-ik+1)} dx + \int_0^{\infty} e^{x(-ik-1)} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{1-ik} + \frac{1}{1+ik} \right] \\ &= \boxed{\frac{2}{\pi} \frac{1}{1+k^2}}\end{aligned}$$

- b. What is the Hermitian conjugate of the operator  $\hat{P} = i$  (think of  $\hat{P}$  as a function of  $x$ , and use the integral form of Hermiticity).

$$\begin{aligned}\langle \psi | \hat{P} | \psi \rangle &= \int_{-\infty}^{+\infty} \psi^*(x) \cdot i \psi(x) dx \\ &= - \int_{-\infty}^{+\infty} (i \psi(x))^* \psi(x) dx\end{aligned}$$

Then  $\hat{P}^\dagger = -\hat{P}$

c. The Legendre polynomials are orthonormal:  $\int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \delta_{\ell\ell'}$ . Use this, and the first two polynomials  $P_0(x) = 1$ ,  $P_1(x) = x$  to construct  $P_2(x)$  (hint: Start with the most general quadratic:  $\bar{P}_2(x) = ax^2 + bx + c$  and apply the orthogonality conditions to fix  $(a, b, c)$ ).

$$\int_{-1}^1 \bar{P}_2(x) P_0(x) dx = \int_{-1}^1 (ax^2 + bx + c) dx = \left(\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx\right) \Big|_{x=-1}^{x=1} \\ = \frac{2}{3}c + 2c = 0 \Rightarrow c = -\frac{1}{3}a$$

$$\int_{-1}^1 \bar{P}_2(x) P_1(x) dx = \int_{-1}^1 (ax^2 + bx + c)x dx = \left(\frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2\right) \Big|_{x=-1}^{x=1} \\ = \frac{2}{3}b = 0$$

so we have  $\bar{P}_2(x) = a(x^2 - \frac{1}{3})$

to

$$\int_{-1}^1 \bar{P}_2(x) \bar{P}_2(x) dx = \int_{-1}^1 a^2(x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx = a^2 \left[ \frac{1}{5}x^5 - \frac{2}{3}x^3 + \frac{1}{9}x \right] \Big|_{x=-1}^{x=1} \\ = a^2 \left[ \underbrace{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}}_{\frac{8}{45}} \right]$$

to this is

$$= \frac{2}{5} \Rightarrow a = \frac{3}{2}$$

Then  $P_2(x) = \frac{3}{2}(x^2 - \frac{1}{3})$

**Problem 2.3**

For the following initial wavefunction of Hydrogen<sup>1</sup>:

$$\psi(r, \theta, \phi) = \frac{1}{8a\sqrt{\pi a^3}} r e^{-\frac{r}{2a}} e^{-i\phi} \sin\theta, \quad (2)$$

- a. Write the time-dependent solution  $\Psi(\mathbf{r}, t)$ .

We can find  $E$  directly by solving

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{e^2}{4\pi\epsilon_0 r}\right)\Psi(\vec{r}) = E\Psi(\vec{r})$$

But from its form, this is clearly the  $n=2$  case  
 $(e^{-\frac{r}{2a}} = e^{-\frac{r}{2a}} \text{ in } (2))$ .

$$\Psi(\vec{r}, t) = \psi(r, \theta, \phi) e^{-iEt/\hbar}$$

- b. What values could you get from an energy measurement?

only  $E_2$ .

<sup>1</sup>Note that in terms of the Bohr radius  $a$ , the Hydrogen potential can be written as  $V(r) = -\frac{\hbar^2}{m a r}$ .