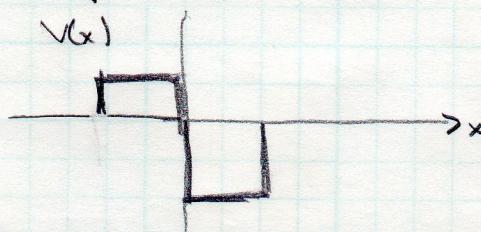


Problem 1

- a. We can clearly see a reflection, indicating a bump; & to the right of that, a resonance, indicating a well:



b. For $f(x) = \sin(\alpha x)$

$$\begin{aligned}\hat{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sin(\alpha x) e^{-ikx} dx \\ &= \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{1}{2i} [e^{i\alpha x} - e^{-i\alpha x}] e^{-ikx} dx \\ &= \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{+\infty} [e^{i(\alpha-k)x} - e^{-i(\alpha+k)x}] dx \quad \text{using } \delta(\alpha-k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(\alpha-k)x} dx \\ &= i\sqrt{\frac{\pi}{2}} [\delta(\alpha-k) - \delta(\alpha+k)]\end{aligned}$$

or $\boxed{\hat{f}(k) = i\sqrt{\frac{\pi}{2}} [\delta(\alpha+k) - \delta(\alpha-k)]}$

c. $\bar{\psi}(x) = Ax(x-a)$ - we can normalize this:

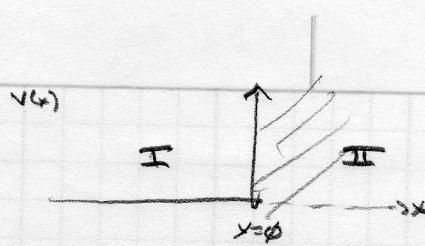
$$\begin{aligned}\int_0^a \bar{\psi}^*(x) \bar{\psi}(x) dx &= A^2 \int_0^a [x^2(x^2 - 2ax + a^2)] dx \\ &= A^2 \left[\frac{1}{5}a^5 - \frac{1}{2}a^5 + \frac{1}{3}a^6 \right] = A^2 a^5 \left[\frac{6}{30} - \frac{15}{30} + \frac{10}{30} \right] \\ &= \frac{A^2 a^5}{30} = 1 \Rightarrow A = \sqrt{\frac{30}{a^5}}.\end{aligned}$$

Then: $\int_0^a \bar{\psi}^* \bar{\psi} dx = \frac{30}{a^5} \int_0^a [x^4 - 2ax^3 + a^2 x^2] dx$

$$\begin{aligned}&= \frac{30}{a^5} \left[\frac{1}{5} \left(\frac{a}{2} \right)^5 - \frac{1}{2} a \left(\frac{a}{2} \right)^4 + \frac{1}{3} a^2 \left(\frac{a}{2} \right)^3 \right] \\ &= \frac{30}{8} \left[\frac{1}{5} \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \right] = \frac{30}{8} \left[\frac{1}{20} - \frac{1}{4} + \frac{1}{3} \right] \\ &= \frac{30}{8} \left[\frac{3}{60} - \frac{15}{60} + \frac{20}{60} \right] = \frac{30}{8} \cdot \frac{8}{60} \\ &= \boxed{1/2}.\end{aligned}$$

Problem 2

$$\text{For } V(x) = \begin{cases} 0 & x < \phi \\ \infty & x > \phi \end{cases}$$



We have 2 regions as shown. In region II, we have $\psi_{\text{II}}(x) = \phi$ (as w/ the infinite square well).

In region I:

$$-\frac{\hbar^2}{2m} \psi''_I(x) = E \psi_I(x) \Rightarrow \psi_I(x) = A e^{ikx} + B e^{-ikx} \quad \text{w/ } k^2 = \frac{2mE}{\hbar^2}$$

The single boundary condition is: $\psi_I(\phi) = \psi_{\text{II}}(\phi)$

$$A + B = \phi \Rightarrow B = -A$$

so

$$\psi(x) = \begin{cases} \bar{A} \sin(kx) & x \leq \phi \\ \phi & x \geq \phi \end{cases}$$

↑ constant to be used in normalization.

$$\rightarrow E_k = \frac{\hbar^2 k^2}{2m}$$

Problem 3

a. For $\Psi(x) = A(\psi_1(x) + 2\psi_2(x))$.

normalization gives: $\bar{\Psi} = \frac{1}{\sqrt{5}}(\psi_1 + 2\psi_2)$.

Then: $\langle H \rangle = \frac{1}{5}E_1 + \frac{4}{5}E_2$

and we obtain E_1 w/ probability $\frac{1}{5}$, E_2 w/ probability $\frac{4}{5}$.

b. $\langle H \rangle = \frac{\pi^2 n^2}{2m a^2} \left(\frac{1}{5} + \frac{16}{5} \right) = \boxed{\frac{17\pi^2 n^2}{10 m a^2}}$

$\langle x \rangle = \int_0^a \Psi(x,t)^* x \Psi(x,t) dx$

$\sim \Psi(x,t) = \frac{1}{\sqrt{5}} [\psi_1 e^{-i(E_1 t)/\hbar} + 2\psi_2 e^{-i(E_2 t)/\hbar}]$
so that

$$\langle x \rangle = \int_0^a \frac{1}{5} [\psi_1(x)^2 \cdot x + 4\psi_2(x)^2 \cdot x + 2\psi_1(x)\psi_2(x) (e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar})] dx$$

+ we need: $\int_0^a \psi_n(x)^2 \times dx = \frac{2}{a} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{n\pi x}{a}\right) dx$

$$= \frac{2}{a} \left[x \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x}{a}\right) \Big|_0^a - \int_0^a \left(\frac{n\pi}{a}\right) \cos\left(\frac{n\pi x}{a}\right) (\sin\left(\frac{n\pi x}{a}\right) + x \cdot \frac{n\pi}{a} \cos\left(\frac{n\pi x}{a}\right)) dx \right]$$

$$= \frac{2}{a} \left[\int_0^a x \cos^2\left(\frac{n\pi x}{a}\right) dx \right] = \frac{2}{a} \left[\int_0^a x (1 - \sin^2\left(\frac{n\pi x}{a}\right)) dx \right]$$

Then we have: $\frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{2}{a} \left[\frac{1}{2}a^2 - \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \right]$



$$\boxed{\frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = a/2}$$

(+) And $\int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) dx = \left(\frac{x}{\pi}\right) \cos\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) \Big|_0^a - \int_0^a \left(\frac{x}{\pi}\right) \cos\left(\frac{\pi x}{a}\right) (\sin\left(\frac{2\pi x}{a}\right) + x \cdot \frac{2\pi}{a} \cos\left(\frac{2\pi x}{a}\right)) dx$

$$= \int_0^a \left(\frac{x}{\pi}\right) \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx + \int_0^a 2x \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx$$

$$\int_0^a \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx = \left(\frac{x}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \Big|_0^a - \int_0^a \left(\frac{x}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) \left(\frac{2\pi}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx$$

$$= -2 \int_0^a \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx \quad (*)$$

$$= -2 \left[-\frac{a}{\pi} \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) \Big|_{x=0}^a - \int_0^a \left(-\frac{a}{\pi}\right) \cos\left(\frac{\pi x}{a}\right) \frac{2\pi}{a} \sin\left(\frac{2\pi x}{a}\right) dx \right]$$

$$= \frac{2a}{\pi} (-1 - 1) + 4 \int_0^a \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx$$

so

$$\boxed{\int_0^a \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx = \frac{4a}{3\pi}}$$

Problem 3 (continued)

and

$$\int_0^a \cos\left(\frac{\pi x}{a}\right) \times \cos\left(\frac{2\pi x}{a}\right) dx = \left(\frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) \times \cos\left(\frac{2\pi x}{a}\right) \right]_0^a - \int_0^a \left(\frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) \right) \left[\frac{2\pi}{a} \sin\left(\frac{2\pi x}{a}\right) x + \cos\left(\frac{2\pi x}{a}\right) \right] dx$$

$$= +2 \left\{ \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) x dx - \frac{a}{\pi} \int_0^a \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx \right\}$$

↓
from (*)

we need:

$$\int_0^a \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx = -\frac{1}{2} \int_0^a \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx = -\frac{1}{2} \cdot \frac{4a}{3\pi} = -\frac{2a}{3\pi}$$

$$= 2 \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) x dx + \frac{2a^2}{3\pi^2}$$

And putting it all together in (+)

$$\int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) dx = \frac{4a^2}{3\pi^2} + 2 \left[2 \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) x dx + \frac{2a^2}{3\pi^2} \right]$$

↓

$$-3 \int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) dx = \frac{8a^2}{3\pi^2}.$$

so

$$\int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) dx = -\frac{8a^2}{9\pi^2}$$

or

$$\frac{3}{a} \int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) dx = -\frac{16a}{9\pi^2}$$

The position expectation value is:

$$\langle x \rangle = \frac{1}{5} \left[\frac{a}{2} + 4 \cdot \frac{a}{2} + 2 \left(-\frac{16a}{9\pi^2} \right) \cdot 2 \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right) \right]$$

$$= \frac{a}{2} - \frac{64a}{45\pi^2} \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)$$

or, using $E_1 = \frac{\pi^2 t^2}{2ma^2}$ & $E_2 = \frac{4\pi^2 k^2}{2ma^2}$

$$\langle x \rangle = \frac{a}{2} - \frac{64a}{45\pi^2} \cos\left[\frac{\pi^2 t}{2ma^2} \cdot 3 + \right]$$

Problem 4

The ground state of the SHO satisfies:

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2x^2\psi = \frac{1}{2}\hbar\omega\psi.$$

Try $\psi(x) = Ae^{\alpha x^2}$, then $\psi' = 2\alpha x\psi$ and $\psi'' = 2\alpha\psi + 4\alpha^2x^2\psi$

$$-\frac{\hbar^2}{2m}[2\alpha\psi + 4\alpha^2x^2] + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\hbar\omega$$

$$-2\frac{\hbar^2\alpha^2}{m}x^2 + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\hbar\omega + \frac{\hbar^2}{m}\alpha$$

$$= 0 \quad \Rightarrow \quad \alpha = \pm \frac{m\omega}{2\hbar}$$

$$\alpha^2 = \frac{1}{4}\frac{m^2\omega^2}{\hbar^2} \Rightarrow \alpha = \pm \left(\frac{m\omega}{2\hbar}\right)^2$$

$$\text{so } \alpha = -\frac{m\omega}{2\hbar} \rightarrow$$

$$\psi(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

to set A, we normalize: $\int_{-\infty}^{+\infty} A^2 e^{-\frac{m\omega}{2\hbar}x^2} dx = A^2 \cdot \sqrt{\frac{\pi}{m\omega}} \cdot \sqrt{4\pi}$

$$\boxed{\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}} \quad (*)$$

This is the starting state for the free particle solution at $t=0$:

$$\psi(x, t) = \int_{-\infty}^{+\infty} \Phi(k) \psi_k(x) e^{-iE_k t/k} dk$$

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} \quad , \quad E_k = \frac{k^2\hbar^2}{2m} \rightarrow$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \psi(x) dx \quad \text{w/ } \psi(x) \text{ given by } (*)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left[\frac{m\omega}{2\hbar}x^2 + ikx\right]} dx$$

$$\text{using: } \int_{-\infty}^{+\infty} e^{-(Ax^2+Bx+C)} dx = \sqrt{\frac{\pi}{A}} e^{-C + \frac{B^2}{4A}}, \text{ we have}$$

$$\Phi(k) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{\pi}{m\omega}} e^{-\frac{k^2}{2m\omega}}$$

$$\boxed{\Phi(k) = \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar}\right)^{1/4} e^{-\frac{k^2}{2m\omega}}} \quad (*)$$

Problem 4 (continued)

$$\begin{aligned}
 \text{The full solution: } \Psi(x,t) &= \int_{-\infty}^{+\infty} \phi(k) \Psi_k(x) e^{-i E_k t / \hbar} dk \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \int_{-\infty}^{+\infty} e^{-\left[\left(\frac{\hbar}{2m\omega} + \frac{i\hbar}{\pi\hbar} t \right) k^2 + ikx \right]} dk \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \int_{-\infty}^{+\infty} e^{-\left[\frac{\hbar}{2m} \left(\frac{1}{\omega} + i\frac{t}{\hbar} \right) k^2 + ikx \right]} dk \\
 &= \frac{1}{\sqrt{2\pi}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \sqrt{\frac{2\pi\hbar}{m}} \left(\frac{1}{\omega} + i\frac{t}{\hbar} \right)^{-1/2} e^{-\frac{ix^2 m}{2\hbar(\omega+i\frac{t}{\hbar})}} \\
 &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} (1+i\omega t)^{-1/2} e^{-\frac{m\omega x^2}{2\hbar(1+i\omega t)}}
 \end{aligned}$$

If we wait Δt , then turn on the potential $V(x) = \frac{1}{2}m\omega^2 x^2$, our initial state will be $\Psi(x, \Delta t)$.

The probability of measuring the ground state energy is then $|\alpha|^2$ w/

$$\begin{aligned}
 \alpha &= \int_{-\infty}^{+\infty} \Psi_0(x) \cdot \Psi(x, \Delta t) dx \\
 &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} (1+i\omega\Delta t)^{-1/2} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar} x^2 (1 + \frac{1}{1+i\omega\Delta t})} dx
 \end{aligned}$$

Let $y = -\sqrt{\frac{m\omega}{2\hbar}} \left(\frac{2+i\omega\Delta t}{1+i\omega\Delta t} \right)^{1/2} x$, then

$$\begin{aligned}
 &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{1}{(1+i\omega\Delta t)^{1/2}} \sqrt{\frac{1+i\omega\Delta t}{2+i\omega\Delta t}} \cdot \left(\frac{2\hbar}{m\omega} \right)^{1/2} \sqrt{\pi} \\
 &= \sqrt{\frac{1}{1 + \frac{1}{2}i\omega\Delta t}}
 \end{aligned}$$

$$\therefore |\alpha|^2 = \left(\frac{1}{1 + \frac{1}{4}\omega^2\Delta t^2} \right)^{1/2}$$