

Problem 1.1

a. For  $f(x)$ , we have:

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$$

For  $f(x) = \delta(x-a)$ ,

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \delta(x-a) dx \\ &= \boxed{\frac{1}{2\pi} e^{-ika}}\end{aligned}$$

b. Let  $g(x) = \frac{df}{dx}$ , then

$$g''(x) = -g(x)$$

and this has solution:

$$g(x) = Ae^{ix} + Be^{-ix}$$

we integrate to get the final solution:

$$f'(x) = Ae^{ix} + Be^{-ix} \Rightarrow f(x) = \tilde{A}e^{ix} + \tilde{B}e^{-ix} + C$$

our three independent solutions are  $\boxed{e^{ix}, e^{-ix}, 1}$  constant.

c. For  $f(x) = \sqrt{x}$ ,  $f(x+\epsilon) = f(x) + \epsilon \frac{df}{dx}|_x + \dots$

$$\stackrel{\text{def}}{=} \sqrt{x} + \frac{1}{2} \in \frac{1}{\sqrt{x}}$$

let  $x = 100 \rightarrow \epsilon = 1$ , then

$$\begin{aligned}f(101) &\approx 10 + \frac{1}{2} \cdot 1 \cdot \frac{1}{10} \\ &= 10.05\end{aligned}$$

Problem 1.2

a. For  $H = [a + a_- + \frac{1}{2}] \hbar \omega$

If  $|1\psi_n\rangle = E_n |\psi_n\rangle$

then:  $a_+ (H |1\psi_n\rangle) = E_n (a_+ |1\psi_n\rangle)$  (\*)

Now and  $a_+ H = \hbar \omega a_+ + a_+ a_- + \frac{1}{2} \hbar \omega a_+$

$$a_- a_+ - a_+ a_- = 1, \text{ so}$$

$$= \hbar \omega a_+ + (a_- a_+ - 1) + \frac{1}{2} \hbar \omega a_+$$

and then, (\*) becomes:

$$\hbar \omega [a_+ a_- (a_+ |1\psi_n\rangle) - (a_- |1\psi_n\rangle)] + \frac{1}{2} \hbar \omega (a_+ |1\psi_n\rangle) = E_n (a_+ |1\psi_n\rangle)$$

or

$$H (a_+ |1\psi_n\rangle) = (E_n + \hbar \omega) (a_+ |1\psi_n\rangle)$$

so if  $|1\psi_n\rangle$  is an eigenstate w/ eigenvalue  $E_n$ , then  $a_+ |1\psi_n\rangle$  is an eigenstate w/ eigenvalue  $(E_n + \hbar \omega)$

b. we want  $[L_i, r_j]$  w/  $L_i = \epsilon_{ijk} r_k p_k$ , then:

$$\begin{aligned} [L_i, r_j] &= \epsilon_{ijk} r_k p_k r_j - r_j \epsilon_{ijk} r_k p_k \\ &= \epsilon_{ijk} [r_k r_j p_k - r_j r_k p_k] \\ &\quad \text{note that } r_j p_k - p_k r_j = i \hbar \delta_{jk}, \\ &\quad \text{so } r_k r_j p_k = r_k (r_j p_k - i \hbar \delta_{jk}) \\ &= \epsilon_{ijk} [r_k r_j p_k - i \hbar r_j \delta_{jk} - r_j r_k p_k] \\ &= -i \epsilon_{ijk} r_k \end{aligned}$$

we have:  $[L_j, x] = -i \epsilon_{jex} r_e = -i \epsilon_{yex} z = -i \hbar z$

$$\rightarrow \sigma_x^2 \sigma_y^2 \geq \frac{1}{4} |\langle [x, L_j] \rangle|^2 = \frac{\hbar^2}{4} |\langle z \rangle|^2$$

Problem 1.3

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

$$S^2 = (\vec{S}_1 + \vec{S}_2 + \vec{S}_3) \cdot (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)$$

$$= S_1^2 + 2\vec{S}_1 \cdot \vec{S}_2 + 2\vec{S}_1 \cdot \vec{S}_3 + S_2^2 + 2\vec{S}_2 \cdot \vec{S}_3 + S_3^2$$

$$\rightarrow \vec{S}_A \cdot \vec{S}_B = S_{A+} S_{B+} + S_{A+} S_{B-} + S_{A-} S_{B+}$$

$$\text{so, using } S_{A+} = \frac{1}{2}(S_{A+} + S_{A-}) \quad S_{A-} = \frac{1}{2}(S_{A+} - S_{A-}) \rightarrow \text{sum for } S_A,$$

$$\vec{S}_A \cdot \vec{S}_B = \frac{1}{4} [(S_{A+} + S_{A-})(S_{B+} + S_{B-}) - (S_{A+} - S_{A-})(S_{B+} - S_{B-})] + S_{A+} S_{B+}$$

$$= \frac{1}{2} [S_{A+} S_{B+} + S_{B-} S_{A+}] + S_{A+} S_{B+}$$

b

$$S^2 = S_1^2 + S_2^2 + S_3^2 + S_{1+} S_{2-} + S_{1-} S_{2+} + 2S_{1+} S_{2+} + S_{1+} S_{3-} + S_{1-} S_{3+} + 2S_{1+} S_{3+}$$

$$+ S_{2+} S_{3-} + S_{2-} S_{3+} + 2S_{2+} S_{3+}$$

$$\text{Then: } S^2 | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle = \underbrace{\left[ h^2 \left( \frac{3}{4} + \frac{3}{4} + \frac{3}{4} \right) + 2h^2 \cdot \frac{1}{4} + 2h^2 \cdot \frac{1}{4} + 2h^2 \cdot \frac{1}{4} \right]}_{= h^2 \cdot \left( \frac{9}{4} + \frac{6}{4} \right)} | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle$$

$$= h^2 \cdot \frac{15}{4} = h^2 \cdot \frac{3}{2} \left( \frac{5}{2} + 1 \right) \text{ so this is a state w/ } S = \frac{3}{2}.$$

$$\rightarrow S_z | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle = 3 \cdot \frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle \rightarrow S_z = \frac{3}{2}$$

so we can call this state:  $| \frac{3}{2} \frac{3}{2} \rangle = | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle$ ,

$$\text{Now } S_- | \frac{3}{2} \frac{3}{2} \rangle = \sqrt{3} \left[ \frac{3}{2} \left( \frac{3}{2} + 1 \right) - \frac{3}{2} \left( \frac{3}{2} - 1 \right) \right] | \frac{3}{2} \frac{1}{2} \rangle = \sqrt{3} \sqrt{3} | \frac{3}{2} \frac{1}{2} \rangle$$

$$\rightarrow S_- | \frac{3}{2} \frac{3}{2} \rangle = [S_{1-} + S_{2-} + S_{3-}] | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle$$

$$= \sqrt{3} \sqrt{\frac{1}{2} \left( \frac{1}{2} + 1 \right) - \frac{1}{2} \left( \frac{1}{2} - 1 \right)} \underbrace{[| \frac{1}{2} - \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle + | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} - \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle + | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} - \frac{1}{2} \rangle]}_{= 1}$$

$$\text{so } | \frac{3}{2} \frac{1}{2} \rangle = \frac{1}{\sqrt{3}} [ | \frac{1}{2} - \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle + | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} - \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle + | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} \frac{1}{2} \rangle | \frac{1}{2} - \frac{1}{2} \rangle ]$$

Problem 1.3 (continued)

From the G-G table for combinations of spin 1  $\otimes$  spin  $1/2$ , we have:

$$|1\frac{1}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}[|11\rangle|1\frac{1}{2}-\frac{1}{2}\rangle + \sqrt{2}|10\rangle|1\frac{1}{2}\frac{1}{2}\rangle]$$

we are viewing  $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$  w/  $\vec{S}_1 + \vec{S}_2$  making up the  $|11\rangle + |10\rangle$  portion (i.e. forming a triplet), so using the  $1+1/2$  table.

In terms of  $S_1^2 + S_2^2$  eigenstates:

$$|11\rangle = \begin{matrix} |1\frac{1}{2}\frac{1}{2}\rangle \\ S_1 S_{1z} \end{matrix} |1\frac{1}{2}\frac{1}{2}\rangle \rightarrow |10\rangle = \frac{1}{\sqrt{2}}(|1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}-\frac{1}{2}\rangle + |1\frac{1}{2}-\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle)$$

then, inputting this above, we have:

$$\begin{aligned} |1\frac{1}{2}\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}[(|1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle)|1\frac{1}{2}-\frac{1}{2}\rangle + (|1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}-\frac{1}{2}\rangle + |1\frac{1}{2}-\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle)|1\frac{1}{2}\frac{1}{2}\rangle] \\ &= \frac{1}{\sqrt{3}}[|1\frac{1}{2}-\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle + |1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}-\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle + |1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}\frac{1}{2}\rangle|1\frac{1}{2}-\frac{1}{2}\rangle] \end{aligned}$$

as before

### Problem 1.4

Using:  $a_{\pm} = \frac{1}{\sqrt{2m\hbar\omega}} (\mp i\hbar + m\omega x)$

$$\text{we have: } \hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 + V_0 = \hbar\omega(a_+ a_- + \frac{1}{2}) + V_0$$

& we still have a ground state  $\exists a_- \Psi_0 = 0$ , so

$$\hat{H} \Psi_0 = (\frac{1}{2}\hbar\omega + V_0) \Psi_0$$

& our ground state energy is now  $\frac{1}{2}\hbar\omega + V_0$ , the energy of an excited state is

$$E_n = \frac{1}{2}\hbar\omega + V_0 + n\hbar\omega$$

### Problem 1.5

For an infinite square well w/ no perturbation:

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

we know, from 1<sup>st</sup> order, nondegenerate perturbation theory, that w/ the perturbation in place:

$$E_n \rightarrow E_n + \epsilon \langle \Psi_n | x | \Psi_n \rangle$$

so we need to evaluate the integral:

$$\begin{aligned} \langle \Psi_n | x | \Psi_n \rangle &= \int_0^a \frac{2}{a} \sin^2\left(\frac{n\pi x}{a}\right) x dx \\ &= \frac{2}{a} \int_0^a \sin\left(\frac{n\pi x}{a}\right) x \sin\left(\frac{n\pi x}{a}\right) dx \\ &= \frac{2}{a} \left[ -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{n\pi x}{a}\right) \right]_0^a - \int_0^a \left( -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \times \left( \frac{n\pi}{a} \right) \cos\left(\frac{n\pi x}{a}\right) \right) dx \\ &= \frac{2}{a} \cdot \int_0^a \cos^2\left(\frac{n\pi x}{a}\right) dx = \frac{2}{a} \left[ a - \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \right]. \end{aligned}$$

$$\therefore \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2}$$

$$\therefore E_n \rightarrow \boxed{E_n + \epsilon \frac{a}{2}}$$

Problem 1.6

a. All of these states share the same energy:

$$E_n = -\frac{13.6 \text{ eV}}{n^2} \quad \text{with } n=2, \quad E_2 = -\frac{13.6 \text{ eV}}{4}$$

b. The radial dependence depends on  $n$  &  $l$ ,

$$R_{nl} \sim e^{-\gamma_{2a}(\frac{r}{a})^l} L_{n-l-1}^{2l+1}(\frac{2r}{a})$$

w/  $n=2$ , we have:

$$R_{20} \sim e^{-\gamma_{2a}(\frac{r}{a})^0} L_{1-0}^{2l+1}(\frac{r}{a})$$

For  $l=0$ , we have  $L'_0(x) = -2x + 4$  (all we need is 1<sup>st</sup> order in  $x$ )

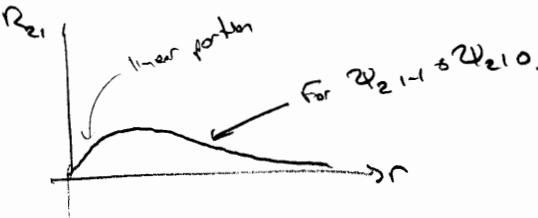
& for  $l=1$ ,  $L'_1 = 6$  (constant)

$$\therefore R_{20} \sim e^{-\gamma_{2a}(2-\frac{r}{a})}$$



And

$$R_{21} \sim e^{-\gamma_{2a}(\frac{r}{a})^1}$$



c. We have  $\Delta E = E_2 - E_3 = -13.6 \text{ eV}(\frac{1}{4} - \frac{1}{9})$  for all three states,

$$= -\frac{5}{36} \cdot 13.6 \text{ eV}$$

$$\Rightarrow \Delta E = h\nu \Rightarrow \nu = \frac{-\frac{5}{36}(13.6 \text{ eV})}{h}$$

d. An equal admixture at time  $t=0$ :

$$\Psi(x, \phi) = \frac{1}{\sqrt{3}} [\psi_{21-1}(x, \phi) + \psi_{210}(x, \phi) + \psi_{200}(x, \phi)]$$

all three states have the same energy, hence, time evolution, so:

$$\Psi(r, \theta, \phi, t) = \frac{1}{\sqrt{3}} [\psi_{21-1}(r, \theta, \phi) + \psi_{210}(r, \theta, \phi) + \psi_{200}(r, \theta, \phi)] e^{i\frac{\epsilon_2}{\hbar}t}$$

$$\text{w/ } E_2 = -\frac{13.6 \text{ eV}}{4}$$