Problem 1.1

a. $a = 5+3i$ has $|a|^2 = a^*a = 25+9 = 34$, so $\boxed{|a| = \sqrt{34}}$

b. Given $f(x) = x(x-1)$, we want $\{\alpha_n\}_{n=1}^{\infty}$ in

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) \qquad (*)$$

the orthogonality of sine is expressed:

$$\int_0^1 \sqrt{2}\sin(n\pi x)\sqrt{2}\sin(m\pi x)\,dx = \delta_{mn}$$

so:

$$2\int_0^1 f(x)\cdot\sin(m\pi x) = \alpha_m \quad \text{(hitting both sides of } (*) \text{ w/ } 2\int_0^1 \sin(m\pi x)\,dx\text{)}$$

& $\int_0^1 f(x)\sin(m\pi x) = \int_0^1 x^2\sin(m\pi x)\,dx - \int_0^1 x\sin(m\pi x)\,dx$

$$= \left[-\frac{1}{m\pi}x^2\cos(m\pi x)\Big|_0^1 + \frac{1}{m\pi}\cdot 2\int_0^1 x\cos(m\pi x)\,dx\right] - \left[-\frac{1}{m\pi}x\cos(m\pi x)\Big|_0^1 + \frac{1}{m\pi}\int_0^1 \cos(m\pi x)\,dx\right]$$

$$= \frac{1}{m\pi}\Big[\underbrace{-\cos(m\pi)}_{\times} + \frac{2}{m\pi}x\cancel{\sin(m\pi x)}\Big|_0^1 - \frac{2}{m\pi}\int_0^1 \sin(m\pi x)\,dx\Big] - \frac{1}{m\pi}\Big[\underbrace{-\cos(m\pi)}_{\times} + \frac{1}{m\pi}\cancel{\sin(m\pi)}\Big]$$

$$= +\frac{2}{(m\pi)^3}\left[\cos(m\pi) - 1\right]$$

$$= +\frac{2}{(m\pi)^3}\left((-1)^m - 1\right).$$

& then: $\boxed{\alpha_m = +\frac{4}{(m\pi)^3}\left((-1)^m - 1\right).}$

c. We know that $|\tfrac{5}{2}\tfrac{5}{2}\rangle = |2\,2\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$ &:

$$S_-|\tfrac{5}{2}\tfrac{5}{2}\rangle = \hbar\sqrt{\tfrac{5}{2}\cdot\tfrac{7}{2} - \tfrac{5}{2}\cdot\tfrac{3}{2}}\,|\tfrac{5}{2}\tfrac{3}{2}\rangle = \sqrt{5}\,\hbar\,|\tfrac{5}{2}\tfrac{3}{2}\rangle.$$

& $S_- = {}^1S_- + {}^2S_-$, so:

$$S_-|\tfrac{5}{2}\tfrac{5}{2}\rangle = ({}^1S_-|2\,2\rangle)|\tfrac{1}{2}\tfrac{1}{2}\rangle + |2\,2\rangle({}^2S_-|\tfrac{1}{2}\tfrac{1}{2}\rangle)$$

$$= \left(\hbar\sqrt{2\cdot 3 - 2\cdot 1}\,|2\,1\rangle\right)|\tfrac{1}{2}\tfrac{1}{2}\rangle + |2\,2\rangle\left(\hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}(-\tfrac{1}{2})}\,|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right)$$

$$= \hbar\cdot 2|2\,1\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \hbar|2\,2\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle.$$

So we have: $2\hbar|2\,1\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \hbar|2\,2\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = \sqrt{5}\,\hbar\,|\tfrac{5}{2}\tfrac{3}{2}\rangle$

or

$$\boxed{|\tfrac{5}{2}\tfrac{3}{2}\rangle = \frac{1}{\sqrt{5}}\left[2|2\,1\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |2\,2\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right].}$$

Problem 1.2

a. $L_z = xp_y - yp_x$ & $[L_z, p_y] = [xp_y, p_y] - [yp_x, p_y]$

$$= \underbrace{xp_yp_y - p_yxp_y}_{=\emptyset} - yp_xp_y + p_yyp_x$$

$$= p_x[p_y, y]$$

$$= \boxed{-i\hbar p_x}$$

b. $\sigma_{L_z}\sigma_{p_y} \geq \frac{1}{2i} i\hbar\langle p_x\rangle = \boxed{\frac{\hbar}{2}\langle p_x\rangle}$

c. Assume $f(x)$ is an eigenfunction of P w/ e-val α:

$$Pf(x) = \alpha f(x)$$

then

$$P^2 f(x) = \alpha^2 f(x)$$

but $P^2 f(x) = Pf(-x) = f(x)$ so

$$f(x) = \alpha^2 f(x) \Rightarrow \boxed{\alpha = \pm 1.}$$

Problem 1.3

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + V_0$

&

$$a_+a_- = \frac{1}{2\hbar m\omega}[-ip + m\omega x][ip + m\omega x] = \frac{1}{2\hbar m\omega}[p^2 - im\omega px + im\omega xp + m^2\omega^2x^2]$$

$$= \frac{1}{2\hbar m\omega}[p^2 + im\omega\underbrace{[x,p]}_{=i\hbar} + m^2\omega^2x^2]$$

$$= \frac{1}{\hbar\omega}\left[\frac{p^2}{2m} - \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2x^2\right]$$

so $a_+a_- = \frac{1}{\hbar\omega}[H - V_0 - \frac{1}{2}\hbar\omega] \Rightarrow H = \hbar\omega a_+a_- + V_0 + \frac{1}{2}\hbar\omega$

Acting on the ground state, ψ_0 w/ $a_-\psi_0 = \emptyset$:

$$H\psi_0 = (V_0 + \tfrac{1}{2}\hbar\omega)\psi_0$$

so $E_0 = \frac{1}{2}\hbar\omega + V_0$ – the ladder still functions, adding $\hbar\omega$ energy at each "rung"

$$\boxed{E_n = (n + \tfrac{1}{2})\hbar\omega + V_0.}$$

Problem 1.4

a. The orbital angular momentum is $l=1$, so ψ could be any energy eigenstate w/ $n \geq 2$, the minimum value would be at $n=2$, so

$$\boxed{E_{min} = \frac{-13.6\,eV}{4}}$$

b. The coefficient in front of χ_+ is: $\frac{1}{\sqrt{3}}$, so the probability of measuring the spin of the e^- at $\hbar/2$ is $\boxed{\frac{1}{3}} = (\frac{1}{\sqrt{3}})^2$.

c. $J_z = L_z + S_z$ so we just need the expectation values for L_z & S_z. Note that:

$$L_z \psi = F(r)\frac{1}{\sqrt{3}}\left[\hbar \cdot \emptyset \, Y_1^0 \chi_+ + \sqrt{2}\, \hbar \cdot 1\, Y_1^1 \chi_-\right]$$

& then

$$\langle L_z \rangle = \frac{2}{3}\hbar$$

For the spin:

$$S_z \psi = F(r)\frac{1}{\sqrt{3}}\left[Y_1^0 \frac{\hbar}{2}\chi_+ + \sqrt{2}\, Y_1^1 \left(-\frac{\hbar}{2}\right)\chi_-\right]$$

so that

$$\langle S_z \rangle = \frac{1}{3}\left[\hbar/2 + 2(-\hbar/2)\right] = -\frac{1}{3}\frac{\hbar}{2}$$

& $$\langle J_z \rangle = \frac{2}{3}\hbar - \frac{1}{3}\cdot\frac{\hbar}{2} = \boxed{\frac{1}{2}\hbar}$$

Problem 1.5

a. The stationary single particle states are: $\psi_{n\ell}(x,y) = \frac{2}{a}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{\ell\pi y}{a}\right)$ w/ $E_{n\ell} = \frac{\pi^2\hbar^2}{2ma^2}(n^2+\ell^2)$

Then the 2 particle states are:

$$\psi^{\pm}_{\substack{n\ell \\ n'\ell'}}(x_1,y_1,x_2,y_2) = A\left[\psi_{n\ell}(x_1,y_1)\psi_{n'\ell'}(x_2,y_2) \pm \psi_{n\ell}(x_2,y_2)\psi_{n'\ell'}(x_1,y_1)\right]$$

w/ energies: $E = \frac{\pi^2\hbar^2}{2ma^2}(n^2+\ell^2+n'^2+\ell'^2)$. The lowest energy state occurs when $n=\ell=n'=\ell'=1$, w/ energy

$$E_{min} = \frac{2\pi^2\hbar^2}{ma^2}$$

the position portion of the ground state is symmetric under particle interchange, so we must be in the anti-symmetric $|00\rangle$ spin state.

$$\psi = \left(\frac{2}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi y_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{\pi y_2}{a}\right)\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right].$$

b. To get $E = 7\frac{\hbar^2\pi^2}{2ma^2}$, we must have $\{n, \ell, n', \ell'\} = \{1, 1, 1, 2\}$.
& we can get this w/:

$$\psi^{\pm}_{\substack{11\\12}},\ \psi^{\pm}_{\substack{11\\21}},\ \psi^{\pm}_{\substack{12\\11}},\ \psi^{\pm}_{\substack{21\\11}}$$

where the − (antisymmetric) position states go w/ the three symmetric spin states, & the + (symmetric) position states go w/ the single anti-symmetric spin state. So, the counting is:

$$3\cdot 4 + 1\cdot 4 = 16 \text{ states w/ energy } E = \frac{7\hbar^2\pi^2}{2ma^2}.$$