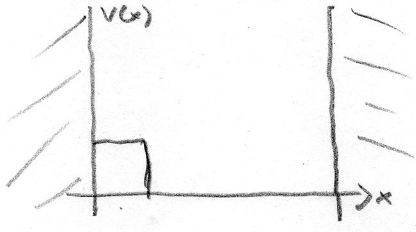
Problem 1.1

a. For the attached movie, sketch a plausible potential, describe your motivation. There are infinite walls on the left and right indicated in blue (?)

The potential must be a barrier, of some sort, to the left of the initial peek in the probability density – the packet spreads out, but there is little probability of finding it in the first ~ 1/4 of the box



b. Mark the (approximate position of the) point $z = i^i$ in the complex plane shown below. The unit circle is shown for reference.

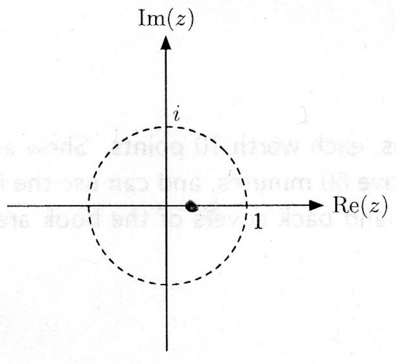


Figure 1: Unit circle in the complex plane.

$i = e^{i\pi/2}$, so $i^i = (e^{i\pi/2})^i = e^{i \cdot i \cdot \pi/2} = e^{-\pi/2}$ & $e > 1$, $\pi/2 > 1$, so

$e^{-\pi/2} < 1$

c. You are given the initial state $\bar{\psi}(x) = A\,x\,(x-a)$ for the infinite square well potential, with $V(0) = V(a) = 0$. Find the probability that a particle is initially between $x = 0$ and $x = \frac{1}{2}\,a$.

First we normalize: $\int_0^a \Psi(x)^* \Psi(x)\,dx = 1$

$$\int_0^a A^2 x^2(x^2 - 2ax + a^2)\,dx = A^2 \int_0^a \left[x^4 - 2ax^3 + a^2x^2\right]dx$$

$$= A^2\left[\tfrac{1}{5}a^5 - \tfrac{2}{4}a^5 + \tfrac{1}{3}a^5\right]$$

$$= A^2 a^5\left[\tfrac{12}{60} - \tfrac{30}{60} + \tfrac{20}{60}\right]$$

$$= \tfrac{1}{30}A^2a^5 = 1 \Rightarrow A = \frac{\sqrt{30}}{a^{5/2}}$$

$$P(0 \to \tfrac{1}{2}a) = \int_0^{\frac{1}{2}a} A^2(x^4 - 2ax^3 + a^2x^2)\,dx$$

$$= \frac{30}{a^5}\left[\frac{1}{5}\frac{a^5}{2^5} - \frac{2}{4}\frac{a^5}{2^4} + \frac{1}{3}\frac{a^5}{2^3}\right]$$

$$= \frac{30}{2^3}\left[\frac{1}{20} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{30}{8}\left[\frac{3}{60} - \frac{15}{60} + \frac{20}{60}\right]$$

$$= \frac{30}{8}\cdot\frac{8}{60}$$

$$= \frac{1}{2}$$

Problem 1.2

Find the stationary states of the infinite step potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ \infty & x > 0 \end{cases} \tag{1}$$

We solve: $-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$ w/ $E > 0$ & $\psi(0) = 0$.

let $k^2 = \frac{2mE}{\hbar^2}$, then $\psi'' = -k^2\psi \Rightarrow \psi(x) = A\cos(kx) + B\sin(kx)$

& $\psi(0) = A = 0 \Rightarrow \psi(x) = B\sin(kx)$ $\quad E(k) = \frac{\hbar^2 k^2}{2m}$

These states are not normalizable.

$$\psi(x) = \begin{cases} B\sin(kx) & x < 0 \\ 0 & x \geq 0 \end{cases} \qquad E(k) = \frac{\hbar^2 k^2}{2m}$$

Problem 3

a. For $\Psi(x) = A(\psi_1(x) + 2\psi_2(x))$.

normalization gives: $\Psi = \frac{1}{\sqrt{5}}(\psi_1 + 2\psi_2)$.

Then: $\langle H \rangle = \frac{1}{5}E_1 + \frac{4}{5}E_2$

and we obtain E_1 w/ probability $1/5$, E_2 w/ probability $4/5$.

b. $\langle H \rangle = \frac{\pi^2\hbar^2}{2ma^2}\left(\frac{1}{5} + \frac{16}{5}\right) = \boxed{\frac{17\pi^2\hbar^2}{10ma^2}}$

$\langle x \rangle = \int_0^a \Psi(x,t)^* x \Psi(x,t)\, dx$

w $\Psi(x,t) = \frac{1}{\sqrt{5}}\left[\psi_1 e^{-iE_1t/\hbar} + 2\psi_2 e^{-iE_2t/\hbar}\right]$

so that

$$\langle x \rangle = \int_0^a \frac{1}{5}\left[\psi_1(x)^2 \cdot x + 4\psi_2(x)^2 x + 2\psi_1(x)\psi_2(x)\left(e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\right)\right] dx$$

+ we need: $\int_0^a \psi_m(x)^2 x\, dx = \frac{2}{a}\int_0^a \underbrace{\sin\left(\frac{m\pi x}{a}\right)}_{v'} \underbrace{x\sin\left(\frac{m\pi x}{a}\right)}_{u} dx$

$$= \frac{2}{a}\left[x\sin\left(\frac{m\pi x}{a}\right)\left(\frac{-a}{m\pi}\right)\cos\left(\frac{m\pi x}{a}\right)\Big|_0^a - \int_0^a \left(\frac{-a}{m\pi}\right)\cos\left(\frac{m\pi x}{a}\right)\left(\sin\left(\frac{m\pi x}{a}\right) + x\cdot\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\right)dx\right]$$

$$= \frac{2}{a}\left[\int_0^a x\cos^2\left(\frac{m\pi x}{a}\right)dx\right] = \frac{2}{a}\left[\int_0^a x\left(1-\sin^2\left(\frac{m\pi x}{a}\right)\right)dx\right]$$

Then we have: $\frac{2}{a}\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)x\,dx = \frac{2}{a}\left[\frac{1}{2}a^2 - \int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx\right]$

$$\Downarrow$$

$$\boxed{\frac{2}{a}\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)x\,dx = a/2}$$

(+) And $\int_0^a \underbrace{\sin\left(\frac{\pi x}{a}\right)}_{v'}\underbrace{x\sin\left(\frac{2\pi x}{a}\right)}_{u}dx = \left(\frac{-a}{\pi}\right)\cos\left(\frac{\pi x}{a}\right)x\sin\left(\frac{2\pi x}{a}\right)\Big|_0^a - \int_0^a\left(\frac{-a}{\pi}\right)\cos\left(\frac{\pi x}{a}\right)\left(\sin\left(\frac{2\pi x}{a}\right) + x\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)\right)dx$

$$= \int_0^a\left(\frac{a}{\pi}\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx + \int_0^a 2x\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx$$

$\int_0^a \underbrace{\cos\left(\frac{\pi x}{a}\right)}_{v'}\underbrace{\sin\left(\frac{2\pi x}{a}\right)}_{u}dx = \left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\Big|_0^a - \int_0^a\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx$

$$= -2\int_0^a \underbrace{\sin\left(\frac{\pi x}{a}\right)}_{v'}\underbrace{\cos\left(\frac{2\pi x}{a}\right)}_{u}dx \qquad (*)$$

$$= -2\left[\frac{-a}{\pi}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\Big|_{x=0}^a - \int_0^a\left(\frac{-a}{\pi}\right)\cos\left(\frac{\pi x}{a}\right)\frac{2\pi}{a}\sin\left(\frac{2\pi x}{a}\right)dx\right]$$

$$= \frac{2a}{\pi}(-1-1) + 4\int_0^a\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx$$

so $\boxed{\int_0^a\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = \frac{4a}{3\pi}}$

Problem 3 (continued)

and

$$\int_0^a \underbrace{\cos\left(\frac{\pi x}{a}\right)}_{v'} \underbrace{x\cos\left(\frac{2\pi x}{a}\right)}_{u} dx = \left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right) x\cos\left(\frac{2\pi x}{a}\right)\Big|_0^a - \int_0^a \left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\left[-\left(\frac{2\pi}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x + \cos\left(\frac{2\pi x}{a}\right)\right]dx$$

$$= +2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx - \frac{a}{\pi}\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx$$

we need:

from (*)

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx = -\frac{1}{2}\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = -\frac{1}{2}\cdot\frac{4a}{3\pi} = \frac{-2a}{3\pi}$$

$$= 2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx + \frac{2a^2}{3\pi^2}$$

And putting it all together in (*)

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{2\pi x}{a}\right)dx = \frac{4a^2}{3\pi^2} + 2\left[2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx + \frac{2a^2}{3\pi^2}\right]$$

$$\Downarrow$$

$$-3\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{2\pi x}{a}\right)dx = \frac{8a^2}{3\pi^2}.$$

so

$$\boxed{\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{2\pi x}{a}\right)dx = \frac{-8a^2}{9\pi^2}}$$

&

$$\boxed{\frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{2\pi x}{a}\right)dx = \frac{-16a}{9\pi^2}}$$

The position expectation value is:

$$\langle x\rangle = \frac{1}{5}\left[\frac{a}{2} + 4\frac{a}{2} + 2\left(\frac{-16a}{9\pi^2}\right)\cdot 2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\right].$$

$$= \frac{a}{2} - \frac{64a}{45\pi^2}\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)$$

or, using $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$ & $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$

$$\boxed{\langle x\rangle = \frac{a}{2} - \frac{64a}{45\pi^2}\cos\left[\frac{\pi^2\hbar}{2ma^2}3t\right]}$$

Problem 4

The ground state of the SHO satisfies:

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\psi = \frac{1}{2}\hbar\omega\psi.$$

Try so $\psi(x) = Ae^{\alpha x^2}$, then $\psi' = 2\alpha x\psi$ & $\psi'' = 2\alpha\psi + 4\alpha^2x^2\psi$

$$-\frac{\hbar^2}{2m}\left[2\alpha\psi + 4\alpha^2x^2\right] + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\hbar\omega$$

$$\underbrace{-2\frac{\hbar^2\alpha^2}{m}x^2 + \frac{1}{2}m\omega^2x^2}_{=\emptyset} = \underbrace{\frac{1}{2}\hbar\omega + \frac{\hbar^2}{m}\alpha}_{=\emptyset \Rightarrow \alpha = -\frac{m\omega}{2\hbar}}$$

$$\Downarrow$$

$$\alpha^2 = \frac{1}{4}\frac{m^2\omega^2}{\hbar^2} \Rightarrow \alpha = \pm\left(\frac{m\omega}{2\hbar}\right)^2.$$

so $\alpha = -\frac{m\omega}{2\hbar}$ &

$$\psi(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

to set A, we normalize: $\int_{-\infty}^{+\infty} A^2 e^{-\frac{m\omega}{\hbar}x^2}\,dx = A^2\cdot\sqrt{\frac{\hbar}{m\omega}}\cdot\sqrt{\pi}$

&

$$\boxed{\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}} \qquad (*)$$

This is the starting state for the free particle solution at $t=\emptyset$,

$$\psi(x,t) = \int_{-\infty}^{+\infty}\Phi(k)\,\psi_k(x)\,e^{-iE_kt/\hbar}\,dk$$

w/ $\psi_k(x) = \frac{1}{\sqrt{2\pi}}$, $E_k = \frac{\hbar^2k^2}{2m}$ &

$$\Phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}\,\psi(x)\,dx \quad \text{w/ } \psi(x) \text{ given by } (*)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left[\frac{m\omega}{2\hbar}x^2 + ikx\right]}\,dx$$

using: $\int_{-\infty}^{+\infty} e^{-(Ax^2+Bx+C)}\,dx = \sqrt{\frac{\pi}{A}}\,e^{-C+B^2/4A}$, we have

$$\Phi(k) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\frac{m\omega}{2\hbar}}}\;e^{-\frac{k^2}{2m\omega/\hbar}}$$

$$\boxed{\Phi(k) = \frac{1}{\sqrt{\pi}}\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} e^{-\frac{k^2\hbar}{2m\omega}}}$$

Problem 4 (continued)

The full solution: $\Psi(x,t) = \int_{-\infty}^{+\infty} \Phi(k)\,\psi_k(x)\,e^{-iE_k t/\hbar}\,dk$

$$= \frac{1}{\sqrt{2}\,\pi}\left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}\int_{-\infty}^{+\infty} e^{-\left[\left(\frac{\hbar}{2m\omega}+\frac{i\hbar}{2m}t\right)k^2+ikx\right]}\,dk$$

$$= \frac{1}{\sqrt{2}\,\pi}\left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}\int_{-\infty}^{+\infty} e^{-\left[\frac{\hbar}{2m}\left(\frac{1}{\omega}+it\right)k^2+ixk\right]}\,dk$$

$$= \frac{1}{\sqrt{2}\,\pi}\left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}\sqrt{\frac{2m\pi}{\hbar}}\left(\frac{1}{\omega}+it\right)^{-1/2} e^{-\frac{+x^2 m}{2\hbar(\omega^{-1}+it)}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(1+i\omega t\right)^{-1/2} e^{-\frac{m\omega x^2}{2\hbar(1+i\omega t)}}$$

If we wait Δt, then turn on the potential $V(x)=\frac{1}{2}m\omega^2x^2$, our initial state will be $\Psi(x,\Delta t)$.

The probability of measuring the ground state energy is then $|a|^2$ w/

$$a = \int_{-\infty}^{+\infty} \psi_0(x)\,\Psi(x,\Delta t)\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(1+i\omega\Delta t\right)^{-1/2}\int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}x^2\left(1+\frac{1}{1+i\omega\Delta t}\right)}\,dx$$

Let $y = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{2+i\omega\Delta t}{1+i\omega\Delta t}\right)^{1/2} x$, then

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{(1+i\omega\Delta t)^{1/2}}\sqrt{\frac{1+i\omega\Delta t}{2+i\omega\Delta t}}\cdot\left(\frac{2\hbar}{m\omega}\right)^{1/2}\sqrt{\pi}$$

$$= \sqrt{\frac{1}{1+\frac{1}{2}i\omega\Delta t}}$$

So $\boxed{|a|^2 = \left(\frac{1}{1+\frac{1}{4}\omega^2\Delta t^2}\right)^{1/2}}$