Problem 2.1

a. Let $f(r) = r^p$, then: $\frac{df}{dr} = pr^{p-1}$ & $\frac{d}{dr}(r^2 f') = \frac{d}{dr}(pr^{p+1}) = p(p+1)r^p$

In order to solve (1):

$$\underbrace{\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)} = 2f(r)$$

$$p(p+1)r^p = 2r^p$$

we need $p=1$, $p=-2$ & our two solutions are:

$$f(r) = r \quad \& \; f(r) = 1/r^2.$$

b. The left sketch is continuous & finite at $r=\varnothing$.

c. The right sketch is continuous, derivative-continuous at R, & is bounded on ~~the~~ right ($s>R$).

d. $\hat{A}^\dagger = \hat{A}$ & $\hat{B}^\dagger = \hat{B}$, then:

$$[\hat{B},\hat{A}]^\dagger = (\hat{B}\hat{A}-\hat{A}\hat{B})^\dagger = \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger = \hat{A}\hat{B}-\hat{B}\hat{A} = [\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$$

& $[\hat{B},\hat{A}]^\dagger = -[\hat{B},\hat{A}]$ means that $[\hat{B},\hat{A}]$ is anti-Hermitian.

e. Take $\hat{Q} = xp$, then: $\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle [H,xp]\rangle = \varnothing$

The commutator is: $[H,xp] = [\frac{p^2}{2m}, xp] + [V(x), xp]$

& we need:

$$[p^2, xp] = ppxp - xppp = p(xp - i\hbar)p - (px + i\hbar)pp$$
$$= pxpp - 2i\hbar p^2 - pxpp$$
$$= -2i\hbar p^2$$

and

$$[V(x), xp]f(x) = V(x)\, x \frac{\hbar}{i}\frac{df}{dx} - x\frac{\hbar}{i}\frac{d}{dx}(V(x)f(x))$$
$$= \cancel{Vx\frac{\hbar}{i}f'} - x\frac{\hbar}{i}(V'f + \cancel{Vf'})$$
$$= +i\hbar x V' \cdot f(x)$$

so $[V(x), xp] = i\hbar x V'$, then

$$[H, xp] = -2i\hbar\frac{p^2}{2m} + i\hbar x V'$$

& $\langle [H,xp]\rangle = -2i\hbar\langle T\rangle + i\hbar\langle xV'\rangle = \varnothing \Rightarrow 2\langle T\rangle = \langle xV'\rangle$

Problem 2.2

We have: $-\frac{\hbar^2}{2m}\nabla^2\psi(\phi) = E\psi(\phi)$

or $\frac{1}{a^2}\frac{d^2\psi}{d\phi^2} = -\frac{2mE}{\hbar^2}\psi$ & let $k^2 \equiv +\frac{2mE}{\hbar^2}$, then

$$\frac{d^2\psi}{d\phi^2} = -k^2a^2\psi$$

& $\psi(\phi) = Ae^{ika\phi} + Be^{-ika\phi}$

If we require that $\psi(\phi+2\pi) = \psi(\phi)$, then

$$Ae^{ika\phi}e^{ika\cdot 2\pi} + Be^{-ika\phi}e^{-ika 2\pi} = Ae^{ika\phi} + Be^{-ika\phi}$$

so we must have $e^{ika\cdot 2\pi} = e^{-ika\cdot 2\pi} = 1 \Rightarrow ka = n$, an integer

Then: $k^2a^2 = n^2$

$$\frac{2mE}{\hbar^2}a^2 = n^2 \Rightarrow E_n = \frac{n^2\hbar^2}{2ma^2}.$$

& $\Delta E = E_2 - E_1 = \frac{3\hbar^2}{2ma^2} = 2\pi\hbar\nu \Rightarrow \nu = \frac{3\hbar}{4\pi m a^2}$

& $\lambda = c/\nu = \frac{4\pi m a^2 c}{\hbar}$

Problem 2.3

Schrödinger's eqn. reads: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\sin(\omega t)\psi = i\hbar\frac{\partial\psi}{\partial t}$ for $\psi(x,t)$

and we have a natural separation of variables here:

$$\psi(x,t) = \psi(x)\Phi(t)$$

w/

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \quad \& \quad i\hbar\dot{\Phi} - V_0\sin(\omega t)\Phi = E\Phi$$

The spatial solution is: $\psi(x) = A\cos(kx) + B\sin(kx)$ w/ $k^2 \equiv \frac{2mE}{\hbar^2}$,
& we have boundary conditions: $\psi(0) = A = 0$, $\psi(a) = B\sin(ka) = 0 \Rightarrow ka = n\pi$
for integer n. Then

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

For the temporal part: $i\hbar\frac{\dot{\Phi}}{\Phi} = V_0\sin(\omega t) + E$

$$\frac{d}{dt}(i\hbar\log(\Phi)) = \frac{d}{dt}\left[-\frac{V_0}{\omega}\cos(\omega t) + Et\right]$$

so

$$\log(\Phi) = -\frac{iEt}{\hbar} + \frac{iV_0}{\hbar\omega}\cos(\omega t) \Rightarrow \Phi(t) = e^{\frac{i}{\hbar}(-Et + \frac{V_0}{\omega}\cos(\omega t))}$$

& $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) \Rightarrow n = 1$, so

$$\psi(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\exp\left[\frac{i}{\hbar}\left(-\frac{\pi^2\hbar^2}{2ma^2}t + \frac{V_0}{\omega}\cos(\omega t)\right)\right]$$