Problem 2.1

(10 points, 3.3 per part)

a. Find the Fourier transform of the function: $f(x) = e^{-|x|}$.[1]

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} e^{-|x|} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x(1-ik)} dx + \int_{0}^{\infty} e^{-x(1+ik)} dx \right]$$

and

$$\int_{-\infty}^{0} e^{x(1-ik)} dx = (1-ik)^{-1} e^{x(1-ik)} \Big|_{x=-\infty}^{0} = (1-ik)^{-1}$$

$$\int_{0}^{\infty} e^{-x(1+ik)} dx = -(1+ik)^{-1} e^{-x(1+ik)} \Big|_{x=0}^{\infty} = (1+ik)^{-1}$$

putting those together:

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-ik} + \frac{1}{1+ik} \right] = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{1+k^2}$$

$$\boxed{F(k) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+k^2}}$$

b. What is the Hermitian conjugate of the operator $\hat{P} = i$ (think of $\hat{P}$ as a function of x, and use the integral form of Hermiticity in one dimension).

$$\langle \psi | P | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \cdot i \, \psi(x) \, dx$$

$$= -\int_{-\infty}^{+\infty} (i\,\psi(x))^* \psi(x) \, dx$$

$$= -\langle P\psi | \psi \rangle = \langle P^\dagger \psi | \psi \rangle$$

so $\boxed{P^\dagger = -P = -i}$

[1]The Fourier transform of a function is defined to be:

$$F(k) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\,k\,x}\, dx. \quad (1)$$

c. The Legendre polynomials are orthonormal: $\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\,\ell+1}\, \delta_{\ell\ell'}$. Use this, and the first two polynomials $P_0(x) = 1$, $P_1(x) = x$ to construct $P_2(x)$ (Hint: Start with the most general quadratic: $\bar{P}_2(x) = a\,x^2 + b\,x + c$ and apply the orthogonality conditions to fix (a, b, c)).

$\bar{P}_2(x) = ax^2 + bx + c$

Impose orthogonality w/ $P_0(x)$:

$$\int_{-1}^{1} P_0(x)\bar{P}_2(x)dx = \emptyset = \int_{-1}^{1} 1\cdot(ax^2+bx+c)dx = \left[\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx\right]_{x=-1}^{1} = \tfrac{2}{3}a + 2c$$

so $c = -\frac{1}{3}a$.

$\bar{P}_2(x) = ax^2 + bx - \frac{1}{3}a$.

Impose orthogonality w/ $P_1(x)$:

$$\int_{-1}^{1} P_1(x)\bar{P}_2(x)dx = \emptyset = \int_{-1}^{1} x\,(ax^2+bx-\tfrac{1}{3}a)dx = \left[\tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 - \tfrac{1}{6}ax^2\right]_{x=-1}^{1} = \tfrac{2}{3}$$

so $b = \emptyset$.

$\bar{P}_2(x) = a(x^2 - \frac{1}{3})$

Finally, we normalize:

$$\int_{-1}^{1} \bar{P}_2(x)\bar{P}_2(x)dx = \frac{2}{2\cdot 2+1} = \int_{-1}^{1} a^2(x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9})dx = a^2\left[\tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x\right]_{x=-1}^{1}$$

$$= a^2\left[\tfrac{2}{5} - \tfrac{4}{9} + \tfrac{2}{9}\right] = a^2\,\tfrac{8}{45}$$

so $a^2 = \frac{45}{8}\cdot\frac{2}{5} = \frac{9}{4} \Rightarrow a = \frac{3}{2}$

$$\boxed{P_2(x) = \tfrac{3}{2}(x^2 - \tfrac{1}{3}) = \tfrac{1}{2}(3x^2 - 1)}$$

Problem 2.2
(10 points)

Find the stationary states and allowed energies for the two-dimensional "infinite rectangular box" potential:

$$V(x,y) = \begin{cases} 0 & 0 < x < a \text{ and } 0 < y < b \\ \infty & \text{otherwise} \end{cases} \tag{2}$$

The stationary states satisfy:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} + \underbrace{V(x,y)}_{\emptyset \text{ inside}}\psi = E\psi \qquad (*)$$

subject to the boundary conditions: $\psi(\emptyset,y) = \psi(a,y) = \psi(x,\emptyset) = \psi(x,b) = \emptyset$.

Use separation of variables: $\psi(x,y) = X(x)\varphi(y)$, then $(*)$ reads

$$\underbrace{-\frac{\hbar^2}{2m}\frac{X''}{X}}_{\equiv\alpha^2} - \underbrace{\frac{\hbar^2}{2m}\frac{\varphi''}{\varphi}}_{\equiv\beta^2} = E.$$

For constants α^2 & β^2. Solving:

$$-\frac{\hbar^2}{2m}\frac{X''}{X} = \alpha^2 \Rightarrow X(x) = A\cos\left(\sqrt{\tfrac{2m\alpha^2}{\hbar^2}}\,x\right) + B\sin\left(\sqrt{\tfrac{2m\alpha^2}{\hbar^2}}\,x\right)$$

& $X(\emptyset) = \emptyset = A$, $X(a) = B\sin\left(\sqrt{\frac{2m\alpha^2}{\hbar^2}}\,a\right) = \emptyset$

$$\Longrightarrow \sqrt{\tfrac{2m\alpha^2}{\hbar^2}}\,a = l\pi \text{ for } l \in \mathbb{Z}.$$

Similarly $\varphi(y) = C\cos\left(\sqrt{\frac{2m\beta^2}{\hbar^2}}\,y\right) + D\sin\left(\sqrt{\frac{2m\beta^2}{\hbar^2}}\,y\right)$
w/ $\varphi(\emptyset) = \emptyset = C$ & $\varphi(b) = \emptyset = D\sin\left(\sqrt{\frac{2m\beta^2}{\hbar^2}}\,b\right) \Rightarrow \sqrt{\frac{2m\beta^2}{\hbar^2}}\,b = n\pi$ for $n \in \mathbb{Z}$.

Then our solution is:

$$\boxed{\psi(x,y) = A\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$$

w/ $\boxed{E_{nl} = \alpha^2 + \beta^2 = \frac{\pi^2\hbar^2}{2m}\left[\frac{l^2}{a^2} + \frac{n^2}{b^2}\right]}$

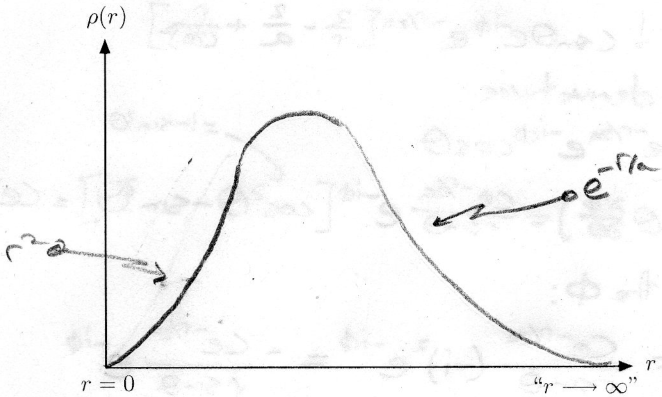
Problem 2.3

(10 = 3+4+3)

For the following initial wavefunction of Hydrogen (this is a normalized stationary state) [2]:

$$\psi(r,\theta,\phi) = \frac{1}{8\,a\,\sqrt{\pi\,a^3}}\,r\,e^{-\frac{r}{2\,a}}\,e^{-i\,\phi}\,\sin\theta, \tag{4}$$

a. Sketch the probability density at $\theta = \frac{\pi}{2}$ as a function of r below – be sure to clearly capture the behavior as $r \longrightarrow 0$ and $r \longrightarrow \infty$ in your sketch:



$\psi^*\psi \sim r^2 e^{-r/a} \sin^2\theta$

& w/ $\theta = \pi/2$,

$\rho(r) \sim r^2 e^{-r/a}$

$\rho \xrightarrow[r\to 0]{} \emptyset$

and

$\rho \xrightarrow[r\to\infty]{} \emptyset$.

b. If you made a measurement of the z-component of angular momentum, what value would you get? (the operator is $L_z \doteq \frac{\hbar}{i}\,\frac{\partial}{\partial\phi}$)

$L_z\psi = \frac{1}{8a\sqrt{\pi a^3}}\, r e^{-r/2a} \sin\theta(-\hbar e^{-i\phi})$

$= -\hbar\psi$

so we will get a measured value of $\boxed{-\hbar.}$

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[2]Note that in terms of the Bohr radius a, the Hydrogen potential can be written as $V(r) = -\frac{\hbar^2}{m\,a\,r}$. For reference, the Laplacian, in spherical coordinates, is:

$$\nabla^2\,\psi(\mathbf{r}) = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\theta}\,\frac{\partial^2\psi}{\partial\phi^2}. \tag{3}$$

c. Write the time-dependent solution $\Psi(\mathbf{r}, t)$.

$$\psi(\vec{r},t) = \psi(\vec{r})\, e^{-iEt/\hbar}$$

we need the energy of the state, & we can get this from $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + \left(\frac{-\hbar^2}{mar}\right)\psi = E\psi \qquad (*)$$

we need the derivatives - let $\psi = C r e^{-r/2a}\sin\theta\, e^{-i\phi}$ w/ $C \equiv \frac{1}{8a\sqrt{\pi a^3}}$.

$$\frac{\partial}{\partial r}\psi = C\sin\theta\, e^{-i\phi}\left[e^{-r/2a} - \frac{r}{2a}e^{-r/2a}\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = C\sin\theta\, e^{-i\phi}\frac{1}{r^2}\left[2re^{-r/2a} - \frac{r^2}{2a}e^{-r/2a} - \frac{3r^2}{2a}e^{-r/2a} + \frac{r^3}{(2a)^2}e^{-r/2a}\right]$$

$$= C\sin\theta\, e^{i\phi}e^{-r/2a}\left[\frac{2}{r} - \frac{2}{a} + \frac{r}{(2a)^2}\right]$$

For the θ-derivatives:

$$\frac{\partial\psi}{\partial\theta} = C r e^{-r/2a}e^{-i\phi}\cos\theta$$

$$\& \ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = \frac{Ce^{-r/2a}}{r\sin\theta}e^{-i\phi}\left[\cos^2\theta - \sin^2\theta\right] = Ce^{-r/2a}e^{-i\phi}\left[\frac{1}{r\sin\theta} - \frac{2}{r}\sin\theta\right]$$

$=1-\sin^2\theta$

& finally, the ϕ:

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{Ce^{-r/2a}}{r\sin\theta}(-i)^2e^{-i\phi} = -\frac{Ce^{-r/2a}}{r\sin\theta}e^{-i\phi}$$

Putting it all together, $(*)$ reads:

$$-\frac{\hbar^2}{2m}e^{-i\phi}e^{-r/2a}\left\{\sin\theta\left[\frac{2}{r} - \frac{2}{a} + \frac{r}{(2a)^2}\right] + \frac{1}{r\sin\theta} - \frac{2}{r}\sin\theta - \frac{1}{r\sin\theta} + \frac{2\sin\theta}{ar}\cdot r\right\} = E r e^{-r/2a}\sin\theta\, e^{-i\phi}$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{1}{(2a)^2} = E} \quad \& \text{ we can write this as: } E = -\frac{\hbar^2}{2ma^2}\cdot\frac{1}{2^2}$$

so this state has $n=2$.

$$\boxed{\psi(\vec{r},t) = \frac{1}{8a}\frac{1}{\sqrt{\pi a^3}}\, r e^{-r/2a}\sin\theta\, e^{-i\phi}\, e^{-i\frac{\hbar}{8ma^2}t}}$$