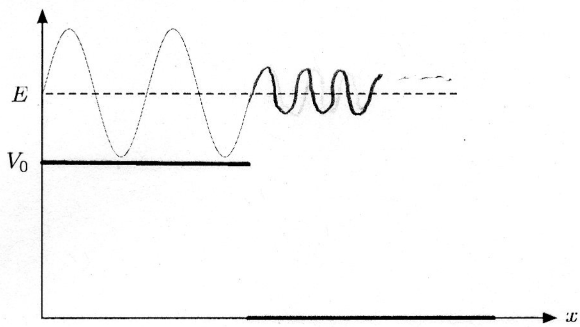
Problem 1.1

(10 points, $3.\bar{3}$ per part)

a. For the step potentials shown below, I have drawn part (of the real part) of a stationary wave function – in each case, complete the sketch by extending the drawing on the right (make sure to reflect plausible wavelength relations and boundary conditions in your sketches, do not worry about the relative amplitude). Note that the energy of the wavefunction is being used as the x-axis for the sketch.

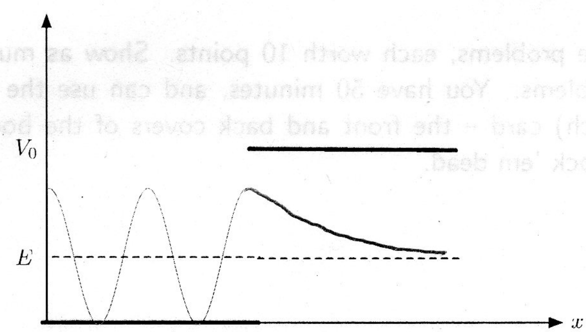


On the left,
$$-\frac{\hbar^2}{2m}\psi'' + V_0\psi = E\psi$$
so
$$\psi'' = -\frac{2m}{\hbar^2}\underbrace{(E-V_0)}_{>0}\psi$$
Let $k_\ell^2 = \frac{2mE}{\hbar^2} - \frac{2mV_0}{\hbar^2}$

then $\lambda_\ell = 2\pi/k_\ell$

On the right, $k_r = \frac{2mE}{\hbar^2} > k_\ell$

so $\lambda_r < \lambda_\ell$.



tunneling

Figure 1: Extend the wave function.

b. Given a stationary state: $\psi(x) = A\,x\,e^{-\alpha\,x^2}$ $(\alpha > 0)$, find the potential $V(x)$ and associated energy (assume that $V(0) = 0$). Be sure to check that both your candidate potential and the energy have the correct units.

$\psi'(x) = Ae^{-\alpha x^2} - 2A\alpha x^2 e^{-\alpha x^2}$ & $\psi''(x) = -2A\alpha x e^{-\alpha x^2} - 4A\alpha x e^{-\alpha x^2} + 4A\alpha^2 x^3 e^{-\alpha x^2}$

$= \underbrace{Axe^{-\alpha x^2}}_{\psi}[-6\alpha + 4\alpha^2 x^2]$

Then Schrödinger's equation gives:

$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi(x) \Rightarrow -\frac{\hbar^2}{2m}\psi[-6\alpha + 4\alpha^2 x^2] + V(x)\psi = E\psi$

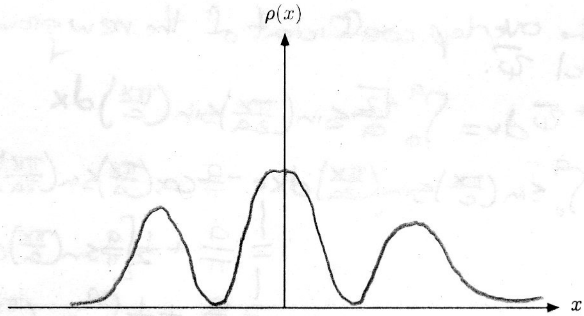
or: $-\frac{2\hbar^2\alpha^2}{m}x^2 + V(x) = E - \frac{3\hbar^2\alpha}{m}$

and we have: $\boxed{V(x) = \frac{2\hbar^2\alpha^2}{m}x^2}$, $\boxed{E = \frac{3\hbar^2\alpha}{m}}$

α has units of $1/m^2$, so $|V| = \frac{J^2 \cdot s^2 \cdot \frac{1}{m^4}}{kg} \cdot m^2 = kg\,m^2/s^2$ ✓

& $|E| = \frac{J^2 \cdot s^2}{kg} \cdot \frac{1}{m^2} = kg\,m^2/s^2$ ✓

c. For a particle in some potential, we construct a state from an admixture of the first five stationary states – the possible measured energies are: $E = 1$ J, $E = 3$ J, $E = 5$ J, $E = 7$ J, and $E = 9$ J (with $E = 1$ J the minimum energy associated with the potential), sketch a plausible probability density for the particle after an energy measurement of 5 J – briefly motivate your sketch.



a measurement of 5J puts the wavefunction in the third energy eigenstate

Roughly:

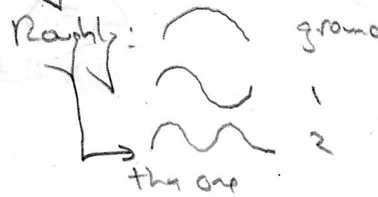


Figure 2: Sketch of probability density after an energy measurement returning 5 J.

Problem 1.2

(10 points)

A particle is in an infinite square well of width a, so that the potential is:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases} \tag{1}$$

We make an energy measurement of $E = \frac{\pi^2 \hbar^2}{2 m a^2}$. Then the original infinite square well is turned off, and (instantaneously), a new infinite square well potential is turned on, this time the well has width $2a$ – what is the new ground state energy, and with what probability will an energy measurement return this ground state energy?

We solve Schrödinger's eqn. w/ $\psi(0)=0$ & $\psi(2a)=0$:

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \Rightarrow \psi(x) = A\cos\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) + B\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right)$$

Then $\psi(0) = A = 0$, &

$\psi(2a) = B\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\cdot 2a\right) = 0 \Rightarrow \sqrt{\frac{2mE}{\hbar^2}}\cdot 2a = n\pi$ for integer n

then: $E_n = \frac{n^2\pi^2\hbar^2}{8ma^2}$ & the ground state energy is: $\boxed{E_1 = \frac{\pi^2\hbar^2}{8ma^2}}$

We are "told" that when the 2nd potential turns on, the state of the particle is given by: $\Psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 \le x \le a$, & 0 for all other values of x.

We want the overlap coefficient of the new ground state: $\psi_1 = \sqrt{\frac{1}{a}}\sin\left(\frac{\pi x}{2a}\right)$ w/ this initial Ψ:

$$c_1 = \int_0^{2a} \psi_1^* \Psi\, dx = \int_0^a \frac{\sqrt{2}}{a}\sin\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)dx$$

Note that: $\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)dx = -\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)\Big|_0^a + \frac{a}{\pi}\frac{\pi}{2a}\int_0^a \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{2a}\right)dx$

$$= \frac{a}{\pi} + \frac{1}{2}\left[\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{2a}\right)\Big|_0^a + \frac{a}{\pi}\frac{\pi}{2a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)dx\right.$$

$$= \frac{a}{\pi} + \frac{1}{4}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)dx$$

So $\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)dx = \frac{4}{3}\frac{a}{\pi}$

and $c_1 = \frac{\sqrt{2}}{a}\cdot\frac{4a}{3\pi} = \frac{4\sqrt{2}}{3\pi}$ – the probability of measuring the particle in the new ground state is $c_1^* c_1 = \frac{16\cdot 2}{9\pi^2} = \boxed{\frac{32}{9\pi^2}}$

Problem 1.3

(10 points, 5 per part)

For the potential (shown in Figure 3)

$$V(x) = \begin{cases} 0 & x < -a \\ V_0 & -a < x < 0 \\ \infty & x > 0 \end{cases} \tag{2}$$

a. Solve Schrödinger's equations for the scattering states with $E > V_0$ for all space (i.e. all values of x) – write your solution in terms of $k^2 \equiv \frac{2mE}{\hbar^2}$ and $\bar{k}^2 \equiv \frac{2m}{\hbar^2}(E - V_0)$. Your solution will involve the usual constants, and you should set up, but not solve, the algebraic equations relating them (so no need to perform a bunch of algebra).

Let $\psi(x) = \begin{cases} \psi_I(x) & x < -a \\ \psi_{II}(x) & -a < x < 0 \\ 0 & x \geq 0. \end{cases}$

In region I: $-\frac{\hbar^2}{2m}\psi_I''(x) = E\psi_I \Rightarrow \boxed{\psi_I(x) = Ae^{ikx} + Be^{-ikx}}$

In region II: $-\frac{\hbar^2}{2m}\psi_{II}'' + V_0\psi_{II} = E\psi_{II} \Rightarrow \psi_{II}'' = -\frac{2m}{\hbar^2}(E - V_0)\psi_{II}$

so $\boxed{\psi_{II}(x) = Ce^{i\bar{k}x} + De^{-i\bar{k}x}}$

we have: $\psi_{II}(0) = 0$ (continuity) so $C + D = 0$, & $\psi_{II}(x) = C(e^{i\bar{k}x} - e^{-i\bar{k}x})$

To find C & B in terms of A, we would use continuity (at $x=-a$)

$$\psi_I(-a) = \psi_{II}(-a)$$

$$\boxed{Ae^{-ika} + Be^{ika} = C(e^{-i\bar{k}a} + De^{i\bar{k}a})}$$

and derivative continuity:

$$\psi_I'(-a) = \psi_{II}'(-a)$$

$$\boxed{ik(Ae^{-ika} - Be^{ika}) = i\bar{k}C(e^{-i\bar{k}a} - e^{i\bar{k}a})}$$

b. Sketch a state with $0 < E < V_0$ – use the line marked E as your x axis.

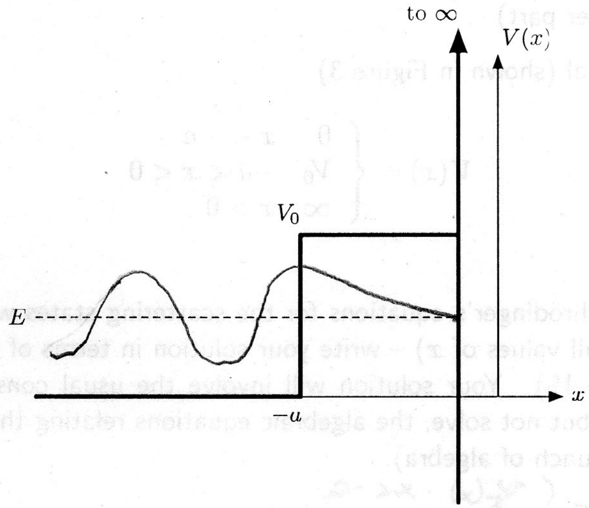


Figure 3: Sketch the wave function using the dashed line as the x axis. Make sure your sketch supports all boundary conditions.

The wavefunction is oscillatory for $x < -a$, then growing & decaying exponentials inside ~~the~~ step, it must go to zero at $x = 0$ to match $\psi(x) = 0$ for $x \geq 0$