Problem 9.1

We have $a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$, $a_- = \frac{1}{\sqrt{2m\hbar\omega}}(ip + m\omega x)$ $\rangle -$ $a_- - a_+ = \sqrt{\frac{2}{m\hbar\omega}}\, ip$.

so $p = -i\sqrt{\frac{m\hbar\omega}{2}}\,(a_- - a_+)$.

then
$$\langle p\rangle = \int_{-\infty}^{+\infty}\psi_j^*\, p\,\psi_j = -i\sqrt{\frac{m\hbar\omega}{2}}\int_{-\infty}^{+\infty}\psi_j^*(a_- - a_+)\psi_j\,dx$$
$$= -i\sqrt{\frac{m\hbar\omega}{2}}\int_{-\infty}^{+\infty}\psi_j^*\left(\sqrt{j}\,\psi_{j-1} - \sqrt{j+1}\,\psi_{j+1}\right)dx = \boxed{\emptyset}$$

(eigenstates are orthogonal).

&
$$\langle p^2\rangle = \int_{-\infty}^{+\infty}\psi_j^*\, p^2\psi_j\,dx = -\frac{m\hbar\omega}{2}\int_{-\infty}^{+\infty}\psi_j^*\left(a_-^2 - a_-a_+ - a_+a_- + a_+^2\right)\psi_j\,dx$$
$$= -\frac{m\hbar\omega}{2}\int_{-\infty}^{+\infty}\psi_j^*\left(\sqrt{j}\sqrt{j-1}\,\psi_{j-2} - (j+1)\psi_j - j\psi_j + \sqrt{j+1}\sqrt{j+2}\,\psi_{j+2}\right)dx$$
$$= \frac{m\hbar\omega}{2}\int_{-\infty}^{+\infty}(2j+1)\,\psi_j^*\psi_j\,dx$$
$$= \boxed{m\hbar\omega(j+1/2)}$$

So $\sigma_p^2 = \langle p^2\rangle - \underbrace{\langle p\rangle^2}_{=\emptyset} = \boxed{m\hbar\omega(j+1/2)}$

b. & $\sigma_x^2 = \frac{E_j}{m\omega^2}$, so $\sigma_x^2\sigma_p^2 = \frac{E_j}{m\omega^2}\cdot m\hbar\omega(j+1/2) = \hbar^2(j+1/2)^2$

& the minimum value occurs at $j=\emptyset$:

$$\boxed{\sigma_x^2\sigma_p^2\Big|_{min} = \frac{\hbar^2}{4}}$$

Problem 9.2

We have: $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$, this is symmetric (as is $\psi^*\psi$) about $x=\emptyset$, so the probability of finding the particle between $\sqrt{2E/m\omega^2}$ & infinity is half the total probability of finding the particle outside the classical region:

$$P = 2\int_{\sqrt{\frac{2E}{m\omega^2}}}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2}dx = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{\sqrt{\frac{2E}{m\omega^2}}}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx$$

$E = \frac{1}{2}\hbar\omega$ for the ground state, so
$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{\sqrt{\frac{\hbar}{m\omega}}}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx$$

let $y \equiv \sqrt{\frac{m\omega}{\hbar}}\,x$, then
$$= \frac{2}{\sqrt{\pi}}\int_1^{\infty} e^{-y^2}dy \approx \boxed{.157}$$

Problem 9.3 (Griffiths 2.13)

We have $\Psi(x,\emptyset) = A[3\psi_0(x) + 4\psi_1(x)]$

a. Our first job is to normalize the initial state:

$$\int_{-\infty}^{+\infty} \Psi(x,\emptyset)^* \Psi(x,\emptyset)dx = A^2 \int_{-\infty}^{+\infty} [9\psi_0^*(x)\psi_0(x) + 24\psi_1^*(x)\psi_0(x) + 16\psi_1^*(x)\psi_1(x)]dx$$

$$\stackrel{!}{=} A^2[9+16] \quad \text{(using orthonormality)}$$

our requirement is that $A^2 25 \stackrel{!}{=} 1$, so $\boxed{A = 1/5.}$

b. We "tack on" the time-dependence for the $\psi_0(x)$ & $\psi_1(x)$ spatial solutions:

$$\Psi(x,t) = \frac{1}{5}[3\psi_0(x)e^{-iE_0t/\hbar} + 4\psi_1(x)e^{-iE_1t/\hbar}]$$

for $E_0 = \frac{1}{2}\hbar\omega$ & $E_1 = \frac{3}{2}\hbar\omega$. Then:

$$\Psi^*\Psi = \frac{1}{25}[9\psi_0(x)^2 + 12\psi_0(x)e^{i(E_0-E_1)t/\hbar}\psi_1(x) + 12\psi_0(x)e^{-i(E_0-E_1)t/\hbar}\psi_1(x) + 16\psi_1(x)^2]$$

$$\stackrel{!}{=} \boxed{\frac{1}{25}[9\psi_0(x)^2 + 24\psi_0(x)\psi_1(x)\cos[(E_0-E_1)t/\hbar] + 16\psi_1(x)^2]},$$

(where I have used the fact that $\psi_0(x)$ & $\psi_1(x)$ are real to omit the *'s).

c. $$\langle x\rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\, x\, \Psi(x,t)dx = \frac{24}{25}\cdot 2\int_0^{\infty}\psi_0(x)\, x\, \psi_1(x)\cos[(E_0-E_1)t/\hbar]\,dx$$

(since $\int_{-\infty}^{+\infty}\psi_0(x)^2 x\,dx = \int_{-\infty}^{+\infty}\psi_1(x)^2 x\,dx = \emptyset$ b/c $x\psi_0^2$ & $x\psi_1^2$ are <u>odd</u>).

$$= \frac{48}{25}\cos(\omega t)\int_0^{\infty}\sqrt{\frac{\hbar}{2m\omega}}\,\psi_0(x)(a_+ + a_-)\psi_1(x)dx$$

$$= \frac{48}{25}\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\int_0^{\infty}[\psi_0(x)\cdot\sqrt{2}\,\psi_2(x) + \psi_0(x)\psi_0(x)]dx$$

orthogonal.

$$= \boxed{\frac{24}{25}\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}}$$

& then $m\frac{d\langle x\rangle}{dt} = \langle p\rangle = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t)$.

d. A measurement of $E_0 = \frac{1}{2}\hbar\omega$ occurs w/ probability $(\frac{3}{5})^2 = \boxed{9/25}$ while a measurement of $E_1 = \frac{3}{2}\hbar\omega$ occurs w/ probability $(4/5)^2 = \boxed{16/25}$. No other energies could be measured.