Problem 8.1

a. $[H,x] = \left[\frac{p^2}{2m}, x\right] + [V,x]$

we know that $[V,x] = \emptyset$ $\quad \left([V,x]T(x) = V(x)\,x\,T(x) - x\,V(x)\,T(x) = \emptyset \right)$

$= \left[\frac{p^2}{2m}, x\right] = \frac{1}{2m}[p^2, x]$

To evaluate this commutator:

$$[p^2, x]\,T(x) = -\hbar^2 \frac{\partial^2}{\partial x^2}\big(x\,T(x)\big) + x\,\hbar^2 \frac{\partial^2}{\partial x^2} T(x)$$

$$= -\hbar^2 \frac{\partial}{\partial x}\left(T + xT'\right) + x\hbar^2 T''$$

$$= -\hbar^2\left(T' + xT'' + T'\right) + x\hbar^2 T''$$

$$= -2\hbar^2 T' = -2\hbar^2 \left(\frac{i}{\hbar}\right) \frac{\hbar}{i}\frac{\partial}{\partial x} T(x)$$

$$= -2\hbar^2 \frac{i}{\hbar}\, pT = -2i\hbar\, pT \quad \text{so} \quad \boxed{[H,x] = \frac{-i\hbar}{m}\,p}$$

we can check this result using: $[x,p] = i\hbar$, $\quad (xp - px = i\hbar)$

$$[p^2, x] = p^2x - xp^2 = p\cdot p\cdot x - x\cdot p\cdot p = p(xp - i\hbar) - (px + i\hbar)p$$

$$= -2i\hbar p$$

b. $[x^n, p] = x^n p - p x^n$

we'll move the p through the x^n to the front:

$$x^n p = x^{n-1} x p = x^{n-1}(px + i\hbar) = x^{n-1} p x + x^{n-1} i\hbar$$

$$= x^{n-2} x p x + x^{n-1} i\hbar = x^{n-2}(px + i\hbar)x + x^{n-1} i\hbar$$

$$= x^{n-2} p x^2 + 2i\hbar x^{n-1}$$

the pattern continues, so that $x^n p = x^{n-j} p x^j + j\, i\hbar x^{n-1}$

let $j = n$, $\quad x^n p = p x^n + n i\hbar x^{n-1}$

so

$$[x^n, p] = x^n p - p x^n = (p x^n + n i\hbar x^{n-1}) - p x^n = \boxed{n i\hbar x^{n-1}}$$

c. $[p, f(x)]\,T(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}(fT) - f\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)T = \frac{\hbar}{i}\left[f'T + fT' - fT'\right] = T\,\frac{\hbar}{i}\frac{\partial}{\partial x} f(x) = T\,pf$

or, dropping the test function: $\boxed{[p, f(x)] = pf(x)}$

using $f(x) = x^n$, we have $\quad pf = \frac{\hbar}{i}\frac{\partial}{\partial x} x^n = -n i\hbar x^{n-1}$ $\quad$ so we confirm:

$$[x^n, p] = -p x^n = +n i\hbar x^{n-1}$$

Problem 8.2 (Griffiths 2.8)

a. We are told that $p(x) = \begin{cases} A & 0<x<a/2 \\ \emptyset & \text{else} \end{cases}$ initially. We can normalize this by requiring:

$$\int_{-\infty}^{+\infty} p(x)\,dx = \int_0^{a/2} A\,dx = A\,a/2 = 1 \Rightarrow A = 2/a$$

Then:

$$\boxed{\Psi_0(x) = \Psi(x,0) = \begin{cases} \sqrt{\frac{2}{a}} & 0<x<a/2 \\ \emptyset & \text{else.} \end{cases}}$$

b. $\frac{\pi^2\hbar^2}{2ma^2} = E_1$ for the infinite square well, so we want $A_1^* A_1$ where A_1 is the coefficient associated w/ $\Psi_1(x)$ in the general solution:

$$\Psi(x,t) = \sum_{j=1}^{\infty} A_j \Psi_j(x) e^{-iE_j t/\hbar}.$$

Then $A_1 = \int_0^a \Psi_1(x)\cdot\Psi_0(x)\,dx = \int_0^{a/2} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\cdot\sqrt{\frac{2}{a}}\,dx = -\frac{2}{a}\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\Big|_{x=0}^{a/2} = \frac{2}{\pi}$

so the prob. of getting E_1 upon energy measurement is $A_1^* A_1 = \boxed{4/\pi^2}$

Problem 8.3

Using: $\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{-iE_n t/\hbar}$

we have:

$$p^2\Psi_n = +\hbar^2\left(\frac{n\pi}{a}\right)^2\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{-iE_n t/\hbar} = +\hbar^2\frac{n^2\pi^2}{a^2}$$

so

$$\langle p^2\rangle = \int_0^a \Psi_n^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\Psi_n(x,t)\,dx = +\hbar^2\frac{n^2\pi^2}{a^2}\underbrace{\int_0^a \Psi_n^*(x,t)\Psi_n(x,t)\,dx}_{=1}$$

$$= \frac{n^2\pi^2\hbar^2}{a^2}$$

so $\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \boxed{\frac{n^2\pi^2\hbar^2}{a^2}}$

Then

$$\sigma_x^2\sigma_p^2 = \left(\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}\right)\left(\frac{n^2\pi^2\hbar^2}{a^2}\right)$$

$$= \frac{n^2\pi^2\hbar^2}{12} - \frac{\hbar^2}{2}$$

$$= \boxed{\frac{\hbar^2}{2}\left[\frac{n^2\pi^2}{6} - 1\right]}$$

The minimum value occurs at $n=1$. $\boxed{\sigma_x^2\sigma_p^2 = \frac{\hbar^2}{2}\underbrace{\left[\frac{\pi^2}{6} - 1\right]}_{\approx .65}}$