Problem 7.1 (Griffiths 2.3)

The time-independent Schrödinger's eqn. reads (for the infinite square well)

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x) \qquad (*)$$

If $E=0$, then $\psi(x)=\alpha x+\beta$. w/ α & β to be set by the boundary conditions ($\psi(0)=\psi(a)=0$):

$$\psi(0)=\beta=0$$

& then

$$\psi(a)=\alpha a = 0 \quad \text{which can only be solved by } \alpha=0$$

then the solution is: $\psi(x)=0$ corresponding to no particle in the well.

If $E<0$, then $(*)$ reads:

$$\frac{\hbar^2}{2m}\psi''(x) = |E|\psi(x) \qquad \text{let } \kappa^2 \equiv \frac{2m|E|}{\hbar^2}$$

the solution is:

$$\psi(x)=\alpha e^{\kappa x}+\beta e^{-\kappa x}$$

& again, α & β can be used to set the boundary conditions:

$$\psi(0)=\alpha+\beta=0 \Rightarrow \beta=-\alpha$$

Then at $x=a$, we require:

$$\psi(a)=\alpha\left(e^{\kappa a}-e^{-\kappa a}\right)=\alpha 2\sinh(\kappa a)=0$$

& this can hold only if $\kappa=0$ or $\alpha=0$, in either case we again get $\psi(x)=0$.

Problem 7.2 (Griffiths 2.5)

a. $$\int_0^a \Psi^*(x,0)\Psi(x,0)\,dx = \int_0^a A^*A\left[\psi_1^*(x)\psi_1(x)+\psi_1^*(x)\psi_2(x)+\psi_1(x)\psi_2^*(x)+\psi_2^*(x)\psi_2(x)\right]dx$$

$$= A^*A\left[1+0+0+1\right]=2A^*A=1 \Rightarrow \boxed{A=1/\sqrt{2}} \text{ (taking } A \text{ real).}$$

b. $\Psi(x,t)$ can be obtained from $\Psi(x,0)$ by decomposing $\Psi(x,0)$ into the spatial basis functions (already done for us) & tacking on the time-dependence. Let $E_n \equiv \frac{n^2\pi^2\hbar^2}{2ma^2}$, then:

$$\Psi(x,t)=\frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-iE_1t/\hbar}+\psi_2(x)e^{-iE_2t/\hbar}\right]$$

then
$$\Psi^*(x,t)\Psi(x,t)=\frac{1}{2}\left[\psi_1 e^{iE_1t/\hbar}+\psi_2 e^{iE_2t/\hbar}\right]\left[\psi_1 e^{-iE_1t/\hbar}+\psi_2 e^{-iE_2t/\hbar}\right]$$

$$=\frac{1}{2}\left[\psi_1^2+\psi_1\psi_2 e^{i(E_1-E_2)t/\hbar}+\psi_2\psi_1 e^{-i(E_1-E_2)t/\hbar}+\psi_2^2\right]$$

$$=\frac{1}{2}\left(\psi_1^2+\psi_2^2\right)+\psi_1\psi_2\cos\left((E_1-E_2)t/\hbar\right)$$

or using $E_1-E_2=\frac{-3\pi^2\hbar^2}{2ma^2}=-3\omega\hbar$,

$$=\boxed{\frac{1}{2}\left(\psi_1^2+\psi_2^2\right)+\psi_1\psi_2\cos(3\omega t)}$$

Problem 7.2 (continued)

c. $\langle x\rangle = \int_0^a \Psi^*(x,t)\, x\, \Psi(x,t)\, dx = \int_0^a \left[\frac{1}{2}\psi_1^2 x + \frac{1}{2}\psi_2^2 x + \psi_1\psi_2 x\cos(3\omega t)\right] dx$

the first two terms are $\frac{1}{2}\frac{a}{2}$ upon integration (we know that $\int_0^a \psi_n^2(x)\, x\, dx = a/2$ for any n from previous problems).

For the third term, we use: $\int_0^a \psi_1\psi_2 x\, dx = \frac{-16a}{9\pi^2}$

so

$$\boxed{\langle x\rangle = a/2 - \frac{16a}{9\pi^2}\cos(3\omega t)}$$

the angular freq. is $3\omega = \boxed{\frac{3\pi^2\hbar^2}{2ma^2}}$ and the amplitude is $\boxed{\frac{16a}{9\pi^2}}$

d. $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{48a}{9\pi^2}\sin(3\omega t)\,\omega = \boxed{m\frac{16a}{3\pi^2}\omega\sin(3\omega t)}$

e. You have a $\frac{1}{2}$–$\frac{1}{2}$ (50–50) chance of getting energy E_1 or E_2, the expectation value, then, is

$$\langle H\rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2 = \boxed{\frac{1}{2}(E_1+E_2)}$$ (average).

Problem 7.3 (Griffiths 2.39)

Given $\Psi(x,t) = \sum_{j=1}^{\infty} A_j \sin\left(\frac{j\pi x}{a}\right) e^{-iE_j t/\hbar}$

we want to find T such that $\Psi(x,t+T) = \Psi(x,t)$, so we need, for every j,

$$e^{-i\frac{E_j}{\hbar}(t+T)} = e^{-i\frac{E_j}{\hbar}t} \Rightarrow e^{-i\frac{E_j}{\hbar}t}\cdot e^{-i\frac{E_j}{\hbar}T} = e^{-i\frac{E_j}{\hbar}t}\cdot 1$$

we must have:

$$e^{-i\frac{E_j}{\hbar}T} = 1 \text{ for all } j \text{ – but } 1 = e^{-i2\pi k} \text{ for integer } k.$$

so we require: $2\pi k = \frac{j^2\pi^2\hbar}{2ma^2}T \Rightarrow T = \frac{4ma^2 k}{j^2\pi\hbar}$ & note that $k = j^2$ is fine, since $j \in \mathbb{Z} \Rightarrow j^2 \in \mathbb{Z}$, so

$$T = \frac{4ma^2}{\pi\hbar} \quad \left(\text{check: } e^{-i\frac{j^2\pi^2\hbar}{2ma^2}\cdot\frac{4ma^2}{\pi\hbar}} = e^{-i2j^2\pi} = 1\ \forall j\ \checkmark\right).$$

b. The classical time is $T = 2a/v$ where v is the speed of the particle – in terms of energy, then: $\boxed{T = \frac{2a}{\sqrt{\frac{2E}{m}}}}$ (since $E = \frac{1}{2}mv^2$)

c. If we take: $\frac{2a}{\sqrt{\frac{2E}{m}}} = \frac{4ma^2}{\pi\hbar} \Rightarrow \frac{4a^2}{\frac{2E}{m}} = \frac{16m^2a^4}{\pi^2\hbar^2} \Rightarrow \frac{m}{2E} = \frac{4m^2a^2}{\pi^2\hbar^2} \Rightarrow \boxed{E = \frac{\pi^2\hbar^2}{8ma^2}}$

$\boxed{\text{Integrals for 7.2}}$

$$\int_0^a \psi_1(x)\, x\, \psi_2(x)\,dx = \frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) x\,dx$$

$$= \frac{2}{a}\left[\cancel{-\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\Big|_{x=0}^{a}} + \frac{a}{\pi}\int_0^a \cos\left(\frac{\pi x}{a}\right)\cdot\left(\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)x + \sin\left(\frac{2\pi x}{a}\right)\right)dx\right]$$

$$= \frac{4}{a}\int_0^a \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)x\,dx + \frac{2}{\pi}\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx \qquad (*)$$

and we need these two integrals:

$$\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = \frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\Big|_{x=0}^{a} - \frac{a}{\pi}\int_0^a \sin\left(\frac{\pi x}{a}\right)\cdot\frac{2\pi}{a}\cos\left(\frac{2\pi x}{a}\right)dx$$

$$= -2\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx$$

$$= -2\left[-\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\Big|_{x=0}^{a} - \frac{a}{\pi}\int_0^a \cos\left(\frac{\pi x}{a}\right)\frac{2\pi}{a}\sin\left(\frac{2\pi x}{a}\right)dx\right]$$

$$= -\frac{4a}{\pi} + 4\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx$$

and gives:

$$-3\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = -\frac{4a}{\pi} \Rightarrow \boxed{\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = +4a/3\pi}$$

For the second integral in (*):

$$\int_0^a \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)x\,dx = \cancel{\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\Big|_{x=0}^{a}} + \frac{a}{\pi}\int_0^a \sin\left(\frac{\pi x}{a}\right)\left[\frac{2\pi}{a}\sin\left(\frac{2\pi x}{a}\right)x - \cos\left(\frac{2\pi x}{a}\right)\right]dx$$

$$= 2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx - \frac{a}{\pi}\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx$$

and now we need:

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx = -\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\Big|_0^a - \frac{a}{\pi}\int_0^a \cos\left(\frac{\pi x}{a}\right)\frac{2\pi}{a}\sin\left(\frac{2\pi x}{a}\right)dx$$

$$= \frac{2a}{\pi} - 2\underbrace{\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx}_{=4a/3\pi}$$

$$= \frac{2a}{\pi} - 8a/3\pi$$

$$= -2a/3\pi$$

$$= 2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx + 2a^2/3\pi^2$$

So in (*) we have:

$$\frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx = \frac{4}{a}\left(2\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx + \frac{2a^2}{3\pi^2}\right) + \frac{2}{\pi}\cdot\frac{4a}{3\pi}$$

or

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx = 4\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx + \frac{4a^2}{3\pi^2} + \frac{4a^2}{3\pi^2}$$

so

$$-3\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx = \frac{8a^2}{3\pi^2}$$

$$\frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)x\,dx = -\frac{16a}{9\pi^2}$$