Problem 5.1 (Griffiths 164)

Given: $\Psi(x,0) = \begin{cases} A\,x/a & 0 \le x \le a \\ A\frac{(b-x)}{b-a} & a \le x \le b \\ 0 & x<0 \text{ or } x>b. \end{cases}$

a. We want $\int_{-\infty}^{+\infty} \Psi^*(x,0)\Psi(x,0)\,dx = 1$

the integral can be restricted to: $x \in [0, b]$

$$\int_{-\infty}^{+\infty} \Psi^*(x,0)\Psi(x,0)\,dx = \int_0^a (A\,x/a)^2 dx + \int_a^b \left(A\,\frac{b-x}{b-a}\right)^2 dx$$

$$= \frac{A^2}{a^2}\cdot\frac{1}{3}a^3 + \frac{A^2}{(b-a)^2}\int_a^b (b^2 - 2xb + x^2)\,dx$$

$$= \frac{1}{3}A^2 a + \frac{A^2}{(b-a)^2}\left[b^2(b-a) - b(b^2-a^2) + \frac{1}{3}(b^3-a^3)\right]$$

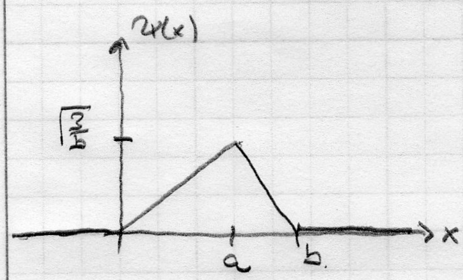
$$= \frac{1}{3}A^2 a + \frac{A^2}{(b-a)^2}\underbrace{\left[\frac{1}{3}b^3 - ab^2 + a^2b - \frac{1}{3}a^3\right]}_{=+\frac{1}{3}(b-a)^3}$$

$$= \frac{1}{3}A^2 a + \frac{1}{3}A^2(b-a)$$

$$= \frac{1}{3}A^2 b$$

and this must be $=1$, so $\frac{1}{3}A^2 b = 1 \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}}$.

b.



c. At $t=0$, the wave function is peaked at $\boxed{a}$, this represents the most likely location of the particle.

d. $P(x>a) = \int_0^a \Psi^*(x,0)\Psi(x,0)\,dx = \int_0^a (A\,x/a)^2 dx = A^2 \cdot \frac{1}{3}a = \boxed{a/b}$

makes sense - if $b=a$, we get 1, if $b=2a$, we get $1/2$.

e. $$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,0)\,x\,\Psi(x,0)\,dx = \int_0^a (A\,x/a)^2 x\,dx + \int_a^b \left(A\,\frac{b-x}{b-a}\right)^2 x\,dx$$

$$= \frac{A^2}{a^2}\frac{1}{4}a^4 + \frac{A^2}{(b-a)^2}\int_a^b x(b^2 - 2xb + x^2)\,dx$$

$$= \frac{1}{4}A^2 a^2 + \frac{A^2}{(b-a)^2}\underbrace{\left[\frac{1}{2}(b^2-a^2)b^2 - \frac{2}{3}(b^3-a^3)b + \frac{1}{4}(b^4-a^4)\right]}_{=\frac{1}{12}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}ba^3 - \frac{1}{4}a^4 = \frac{1}{12}(b-a)^3\cdot(3a+b)}$$

$$= \frac{1}{4}A^2a^2 + \frac{1}{12}A^2(b-a)(3a+b) = A^2\left[\frac{1}{4}a^2 + \frac{1}{12}b^2 - \frac{1}{12}ab - \frac{1}{4}a^2 + \frac{1}{4}ab\right]$$

$$= \frac{3}{b}\left[\frac{1}{12}b^2 + \frac{1}{6}ab\right]$$

$$= \boxed{\frac{1}{4}b + \frac{1}{2}a}$$

Problem 5.2 (Griffiths 1.7)

We have: $\langle p \rangle = \int_{-\infty}^{+\infty} \Psi^* \frac{\hbar}{i}\frac{\partial \Psi}{\partial x} dx$

so

$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{+\infty}\left[\frac{\partial \Psi^*}{\partial t}\cdot\frac{\hbar}{i}\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right)\right]dx$$

& from Schrödinger's eqn. we have:

$$\frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right] \quad \& \quad \frac{\partial \Psi^*}{\partial t} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right]$$

so that:

$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{+\infty}\Bigg[\underbrace{\frac{-1}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right)\frac{\hbar}{i}\frac{\partial \Psi}{\partial x}}_{\text{int. by parts}} + \underbrace{\Psi^*\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)\right)}_{\text{int. by parts}}\Bigg]dx$$

$$= \int_{-\infty}^{+\infty}\left[\left(\frac{+\hbar^2}{2m}\frac{\partial \Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2} + V\Psi^*\frac{\partial\Psi}{\partial x}\right) + \frac{\partial \Psi^*}{\partial x}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)\right]dx$$

cancel

$$= \int_{-\infty}^{+\infty}\left[\Psi^* V\frac{\partial\Psi}{\partial x} + \underbrace{\frac{\partial\Psi^*}{\partial x}V\Psi}_{\text{int. by parts}}\right]dx$$

$$= \int_{-\infty}^{+\infty}\left[\Psi^* V\frac{\partial\Psi}{\partial x} - \Psi^*\left(\frac{\partial V}{\partial x}\Psi + V\frac{\partial \Psi}{\partial x}\right)\right]dx$$

cancel

$$= -\int_{-\infty}^{+\infty}\Psi^*\frac{\partial V}{\partial x}\Psi\, dx$$

and this is what we would call $\left\langle -\frac{dV}{dx}\right\rangle$, so $\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$

Problem 5.3

a. The time-independent Schrödinger eqn. for $V=0$ is:

$$-\frac{\hbar^2}{2m}\Psi'' = E\Psi \Rightarrow \Psi(x) = A\cos\left(\sqrt{\tfrac{2mE}{\hbar^2}}\,x\right) + B\sin\left(\sqrt{\tfrac{2mE}{\hbar^2}}\,x\right)$$

& we require: $\Psi(0) = \boxed{A = 0}$ & $\Psi(d) = B\sin\left(\sqrt{\tfrac{2mE}{\hbar^2}}\,d\right) = 0$

so that

$$\sqrt{\tfrac{2mE}{\hbar^2}}\,d = n\pi \text{ for integer } n \Longrightarrow \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2md^2}}$$

& then $\Psi_n(x) = B\sin\left(\frac{n\pi x}{d}\right)$ w/ $E_n = \frac{n^2\pi^2\hbar^2}{2md^2}$ so the temporal part of the solution is

$$\Phi_n = e^{-iE_nt/\hbar} = e^{-\frac{in^2\pi^2\hbar}{2md^2}t}$$

The full solution w/ energy E_n is: $\Psi_n(x,t) = B\sin\left(\frac{n\pi x}{d}\right)e^{-\frac{in^2\pi^2\hbar}{2md^2}t}$

Problem 5.3 (continued)

b. $\Psi_n(x,t)^* \Psi_n(x,t) = B^2 \sin^2\left(\frac{n\pi x}{d}\right)$.

so $\int_0^d B^2 \sin^2\left(\frac{n\pi x}{d}\right) dx = B^2 \, d/2 = 1 \Rightarrow B = \sqrt{\frac{2}{d}}$.

so

$$\boxed{\Psi_n(x,t) = \sqrt{\tfrac{2}{d}} \sin\left(\tfrac{n\pi x}{d}\right) e^{-i\frac{n^2\pi^2\hbar}{2md^2}t}}$$

pure phase, goes away when we take $\Psi_n^*\Psi_n$

c. $\langle x \rangle = \int_0^d \Psi_n^* x \Psi_n dx = \frac{2}{d}\int_0^d \sin^2\left(\frac{n\pi x}{d}\right) x \, dx$

so $\int_0^d \underbrace{\sin\left(\frac{n\pi x}{d}\right)}_{u'} \underbrace{x \sin\left(\frac{n\pi x}{d}\right)}_{v} dx = \frac{-d}{n\pi}\cos\left(\frac{n\pi x}{d}\right) \cancel{\sin\left(\frac{n\pi x}{d}\right)} \Big|_{x=0}^{d} - \left(\frac{-d}{n\pi}\right)\int_0^d \cos\left(\frac{n\pi x}{d}\right)\left(\sin\left(\frac{n\pi x}{d}\right) + \frac{n\pi x}{d}\cos\left(\frac{n\pi x}{d}\right)\right) dx$

$= \frac{d}{n\pi}\int_0^d \cos\left(\frac{n\pi x}{d}\right)\cancel{\sin\left(\frac{n\pi x}{d}\right)} dx + \int_0^d x \cos^2\left(\frac{n\pi x}{d}\right) dx$

so far, then, we have:

$\int_0^d x \sin^2\left(\frac{n\pi x}{d}\right) dx = \int_0^d x\cos^2\left(\frac{n\pi x}{d}\right) dx = \int_0^d x\left(1 - \sin^2\left(\frac{n\pi x}{d}\right)\right) dx$

then

$\int_0^d x \sin^2\left(\frac{n\pi x}{d}\right) dx = \frac{1}{2}d^2 - \int_0^d x \sin^2\left(\frac{n\pi x}{d}\right) dx \Rightarrow \int_0^d x\sin^2\left(\frac{n\pi x}{d}\right) dx = \frac{1}{4}d^2$

so $\langle x \rangle = \frac{2}{d}\int_0^d \sin^2\left(\frac{n\pi x}{d}\right) x \, dx = \frac{2}{d} \, d^2/4 = \frac{1}{2}d. \Rightarrow \boxed{\langle x \rangle = \tfrac{1}{2} d}$

$\langle p \rangle = \int_0^d \Psi_n^* \frac{\hbar}{i}\frac{\partial\Psi}{\partial x} dx = \frac{2}{d}\int_0^d \sin\left(\frac{n\pi x}{d}\right) \cdot \frac{n\pi}{d}\cos\left(\frac{n\pi x}{d}\right) dx = \boxed{\varnothing}$

evidently these states, w/ $\langle p \rangle = \varnothing$, don't "move".

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