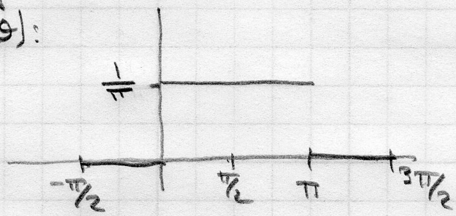
Problem 4.1 (Griffiths 1.4)

a. $p(\theta) = \alpha$, a constant, since we are told that all angles are equally likely, $p(\theta)$ can't depend on angle.

We fix α by normalization: $\int_0^\pi p(\theta)\,d\theta = 1 = \pi\alpha \Rightarrow \alpha = 1/\pi$, so $\boxed{p(\theta) = \begin{cases} \frac{1}{\pi} & \theta \in [0,\pi] \\ 0 & \text{else.} \end{cases}}$

Graph of $p(\theta)$:



b. $\langle\theta\rangle = \int_0^\pi p(\theta)\cdot\theta\,d\theta = \int_0^\pi \frac{1}{\pi}\theta\,d\theta = \frac{1}{\pi}\cdot\frac{1}{2}\pi^2 = \boxed{\pi/2.}$

$\langle\theta^2\rangle = \int_0^\pi p(\theta)\,\theta^2\,d\theta = \int_0^\pi \frac{1}{\pi}\theta^2\,d\theta = \frac{1}{\pi}\frac{1}{3}\pi^3 = \boxed{\pi^2/3.}$

$\sigma = \left[\langle\theta^2\rangle - \langle\theta\rangle^2\right]^{1/2} = \left[\pi^2/3 - (\pi/2)^2\right]^{1/2} = \boxed{\frac{\pi}{2\sqrt{3}}}$

c. $\langle\sin\theta\rangle = \int_0^\pi p(\theta)\sin\theta\,d\theta = \int_0^\pi \frac{1}{\pi}\sin\theta\,d\theta = -\frac{1}{\pi}\cos\theta\Big|_0^\pi = -\frac{1}{\pi}(-1-1) = \boxed{2/\pi.}$

$\langle\cos\theta\rangle = \int_0^\pi p(\theta)\cos\theta\,d\theta = \int_0^\pi \frac{1}{\pi}\cos\theta\,d\theta = \frac{1}{\pi}\sin\theta\Big|_0^\pi = \boxed{0.}$

$\langle\cos^2\theta\rangle = \int_0^\pi p(\theta)\cos^2\theta\,d\theta = \int_0^\pi \frac{1}{\pi}\cos^2\theta\,d\theta = \frac{1}{\pi}\cdot\pi/2 = \boxed{1/2}$

Problem 4.2

a. For $\rho(r) = \begin{cases} Ar & r \le R \\ \varnothing & r > R \end{cases}$

we first normalize the distribution:

$$1 = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(r)\, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 4\pi \int_0^R A r^3 dr$$

$$= 4\pi A \cdot \tfrac{1}{4} R^4$$

So $\boxed{A = \frac{1}{\pi R^4}}$

b. The probability of finding the particle between $\frac{1}{2}R$ & R:

$$P = \int_{\frac{1}{2}R}^{R} \int_0^{2\pi} \int_0^{\pi} \frac{r}{\pi R^4} \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{4}{R^4} \int_{\frac{1}{2}R}^{R} r^3 dr$$

$$= \frac{4}{R^4} \cdot \frac{1}{4}\left[R^4 - \left(\tfrac{1}{2}\right)^4 R^4\right]$$

$$= 1 - \frac{1}{16} = \boxed{15/16}$$

c. $$\langle r \rangle = \int_0^R \int_0^{2\pi} \int_0^{\pi} \frac{r^2}{\pi R^4} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{4}{R^4} \int_0^R r^4 dr$$

$$= \frac{4}{R^4} \cdot \frac{1}{5} R^5$$

$$= \boxed{4/5\, R}$$

Problem 4.3 (Griffiths 1.3)

a. We must have: $$1 = \int_{-\infty}^{+\infty} A e^{-\lambda(x-a)^2} dx \overset{\text{let } y \equiv x-a}{=} A \int_{-\infty}^{+\infty} e^{-\lambda y^2} dy \overset{\text{integrand is sym}}{=} 2A \int_0^{\infty} e^{-\lambda y^2} dy = 2A\sqrt{\pi}\left(\frac{1}{2\sqrt{\lambda}}\right)$$

So $\boxed{A = \sqrt{\frac{\lambda}{\pi}}}$

b. $$\langle x \rangle = \int_{-\infty}^{+\infty} \sqrt{\tfrac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2} x \, dx \overset{y \equiv x-a}{=} \sqrt{\tfrac{\lambda}{\pi}} \int_{-\infty}^{+\infty} e^{-\lambda y^2}(y+a) = \sqrt{\tfrac{\lambda}{\pi}}\left[\underbrace{\int_{-\infty}^{+\infty} y e^{-\lambda y^2} dy}_{\varnothing} + a\int_{-\infty}^{+\infty} e^{-\lambda y^2} dy\right]$$

$$= \underbrace{\left[\sqrt{\tfrac{\lambda}{\pi}} \int_{-\infty}^{+\infty} e^{-\lambda y^2} dy\right]}_{=1} a = \boxed{a}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \sqrt{\tfrac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2} x^2 dx \overset{y \equiv x-a}{=} \sqrt{\tfrac{\lambda}{\pi}} \int_{-\infty}^{+\infty} e^{-\lambda y^2}(y^2 + 2ay + a^2) dy = \sqrt{\tfrac{\lambda}{\pi}} \int_{-\infty}^{+\infty} \Big(e^{-\lambda y^2} y^2 + \underbrace{2ay e^{-\lambda y^2} + a^2 e^{-\lambda y^2}}_{\text{we know those terms already}}\Big) dy$$

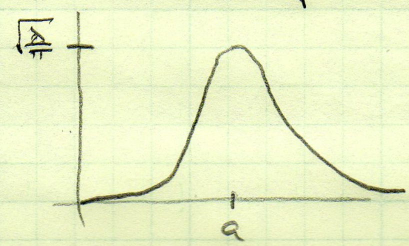
$$= a^2 \underbrace{\int_{-\infty}^{+\infty} \sqrt{\tfrac{\lambda}{\pi}}\, e^{-\lambda y^2} dy}_{=1} + \underbrace{2a\sqrt{\tfrac{\lambda}{\pi}} \int_{-\infty}^{+\infty} y e^{-\lambda y^2} dy}_{=\varnothing} + \int_{-\infty}^{\infty} \sqrt{\tfrac{\lambda}{\pi}}\, y^2 e^{-\lambda y^2} dy$$

$$= a^2 + 2\sqrt{\tfrac{\lambda}{\pi}} \int_0^{\infty} y^2 e^{-\lambda y^2} dy = a^2 + 2\sqrt{\tfrac{\lambda}{\pi}}\, \sqrt{\pi} \cdot 2\left(\frac{1}{2\sqrt{\lambda}}\right)^3 = a^2 + \frac{1}{2\lambda}$$

Problem 4.3 (continued)

then $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda} \Rightarrow \sigma = \frac{1}{\sqrt{2\lambda}}$

c. The Gaussian is peaked around a & decays to zero as $x \to \pm\infty$



optional:

```
In[1]:= NN = 100;
        a = 5.0;
        dx = a / (NN + 1);
        100 N[Sum[Exp[-(j dx)^2] dx, {j, 0, NN}] - (1/2) Sqrt[Pi]]

Out[4]= 2.47525
```

so 2% error

```
In[5]:= NN = 1000;
        a = 5.0;
        dx = a / (NN + 1);
        100 N[Sum[Exp[-(j dx)^2] dx, {j, 0, NN}] - (1/2) Sqrt[Pi]]

Out[8]= 0.24975
```

.2%

```
In[9]:= NN = 10000;
        a = 5.0;
        dx = a / (NN + 1);
        100 N[Sum[Exp[-(j dx)^2] dx, {j, 0, NN}] - (1/2) Sqrt[Pi]]

Out[12]= 0.0249975
```

.02%

```
In[13]:= NN = 25000;
         a = 5.0;
         dx = a / (NN + 1);
         100 N[Sum[Exp[-(j dx)^2] dx, {j, 0, NN}] - (1/2) Sqrt[Pi]]

Out[16]= 0.0099996
```

~ .01%