Problem 34.1. (Griffiths 5.20)

First we'll work on $E > \emptyset$ solutions. We need to solve:

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x) \qquad (x \in (\emptyset, a)) \qquad (*)$$

w/

$$\psi_+(\emptyset) = \psi_-(\emptyset) \qquad \text{(continuity)}$$

$$\psi'_+(\emptyset) - \psi'_-(\emptyset) = -\frac{2m\alpha}{\hbar^2}\psi_+(\emptyset)$$

The solution to $(*)$ is: $\psi(x) = A\sin(kx) + B\cos(kx)$ w/ $k^2 \equiv \frac{2mE}{\hbar^2}$.

Then the solution in the region $(-a, \emptyset)$ is:

$$\underbrace{\psi(x)}_{\in(a,\emptyset)} = e^{-iKa}\underbrace{\psi(x+a)}_{\in(\emptyset,a)} \quad \text{(this is the } \psi_- \text{ we'll use for b.c.)}$$

& we have the continuity condition: $\psi(\emptyset) = e^{-iKa}\psi(a)$ (+)

Derivative discontinuity gives:

$$\psi'(\emptyset) - e^{-iKa}\psi'(a) = -\frac{2m\alpha}{\hbar^2}\psi(\emptyset). \qquad (\circ)$$

(+) & $(\circ)$ read; for our wave-function:

$$B = e^{-iKa}[A\sin(ka) + B\cos(ka)].$$

&

$$kA - e^{-iKa}k[A\cos(ka) - B\sin(ka)] = -\frac{2m\alpha}{\hbar^2}B.$$

rearranging this pair, we get: $A = \frac{B(e^{iKa} - \cos(ka))}{\sin(ka)}$ from the first, & then

$$(e^{iKa} - \cos(ka)) - e^{-iKa}[(e^{iKa} - \cos(ka))\cos(ka) - \sin^2(ka)] = -\frac{2m\alpha}{\hbar^2 k}\sin(ka) \quad \text{inserting in}$$

or:

$$e^{iKa} - \cos(ka) - \cos(ka) + \underbrace{e^{-iKa}\cos^2(ka) + e^{-iKa}\sin^2(ka)}_{= e^{-iKa}} = -\frac{2m\alpha}{\hbar^2 k}\sin(ka)$$

&

$$2\cos(Ka) - 2\cos(ka) = -\frac{2m\alpha}{\hbar^2 k}\sin(ka) \Rightarrow \boxed{\cos(Ka) = \cos(ka) - \frac{\alpha m}{\hbar^2 k}\sin(ka)}$$

& let $z \equiv ka$, $\beta \equiv -\frac{\alpha m a}{\hbar^2}$, then $\cos(Ka) = \cos(z) + \beta\frac{\sin(z)}{z}$.
w/ $\beta < \emptyset$.

Taking $\beta = -1.5$, we get:
(negative portion is also shown)

For the negative solutions, let $\bar{E} \equiv |E|$, then we need to solve:

$$-\frac{\hbar^2}{2m}\psi'' = -\bar{E}\psi \Rightarrow \psi(x) = A\sinh(kx) + B\cosh(kx) \quad w/\ k^2 \equiv \frac{2m\bar{E}}{\hbar^2}$$

imposing the b.c.'s (+) & (*):

$$B = e^{-iKa}\left(A\sinh(ka) + B\cosh(ka)\right).$$

& $kA - e^{-iKa}\cdot k\left[A\cosh(ka) + B\sinh(ka)\right] = -\frac{2m\alpha}{\hbar^2}B.$

solving the first for A: $A = \frac{B\left[e^{+iKa} - \cosh(ka)\right]}{\sinh(ka)}$ & inserting in the second:

$$e^{iKa} - \cosh(ka) - e^{-iKa}\left[\left(e^{iKa} - \cosh(ka)\right)\cosh(ka) + \sinh^2(ka)\right] = -\frac{2m\alpha}{\hbar^2 k}\cdot\sinh(ka)$$

gives:

$$e^{iKa} - \cosh(ka) - \cosh(ka) + \underbrace{e^{-iKa}\cosh^2(ka) - e^{-iKa}\sinh^2(ka)}_{= e^{-iKa}} = -\frac{2m\alpha}{\hbar^2 k}\sinh(ka).$$

then

$$\cos(Ka) - \cosh(ka) = -\frac{m\alpha}{\hbar^2 k}\sinh(ka) \Rightarrow \cos(Ka) = \cosh(ka) - \frac{\alpha m}{\hbar^2 k}\sinh(ka)$$

again, let $z = -ka$, $\beta = +\frac{\alpha m a}{\hbar^2}$, we have:

$$\boxed{\cos(Ka) = \cosh(z) + \beta\frac{\sinh(z)}{z}}$$ now for $z < \emptyset$.

The inclusion of $z < \emptyset$ gives a first band w/

$$\cos(Ka) = 1 \longrightarrow -1,$$

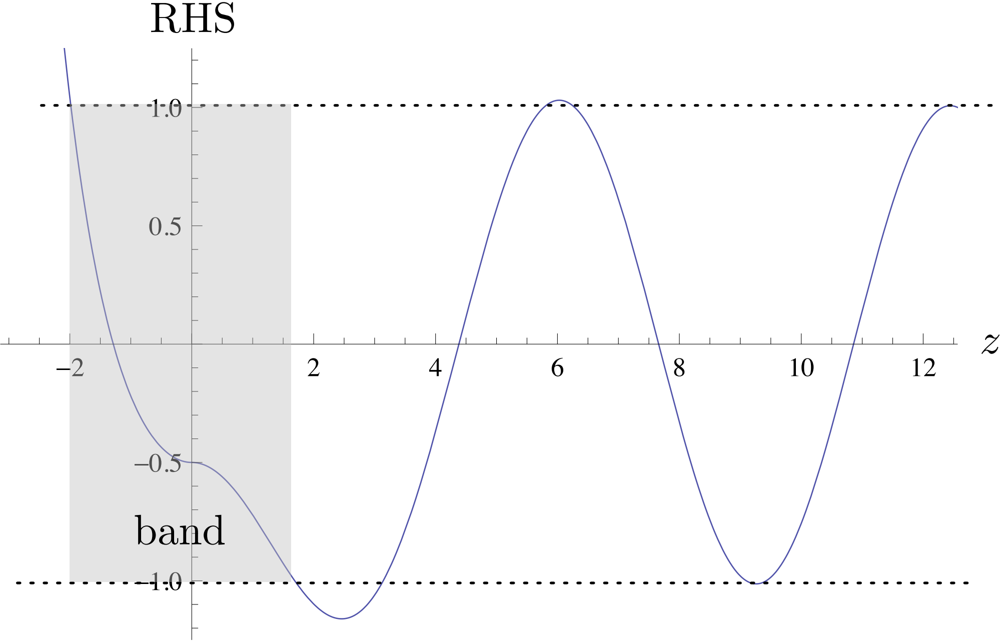
&

$$\cos(Ka) = 1 \Rightarrow Ka = \emptyset \longrightarrow \pi$$

w/ $Ka = \frac{2\pi n}{N} = \emptyset$ for $n = \emptyset$

$= \pi$ for $n = N/2$

So there are $N/2$ states going $1 \to -1$, & another $N/2$ going $-1 \to 1$, for a total of N states in the first band.


RHS
1.0
0.5
−0.5
−1.0
−2
2
4
6
8
10
12
z
band