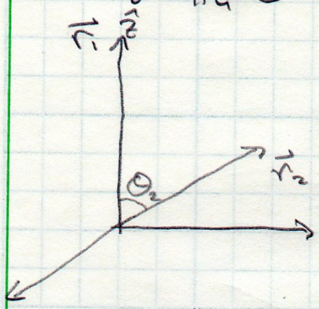
Problem 32.1 (Griffiths 5.11a)

For $\psi_0 = \frac{8}{\pi a^3} e^{-2(r_1+r_2)/a}$, we want to compute $\left\langle \frac{1}{|\vec r_1 - \vec r_2|} \right\rangle$



setting $\hat z$ along $\vec r_1$, we have:

$(\vec r_1 - \vec r_2)\cdot(\vec r_1 - \vec r_2) = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2$

automatically.

Our expectation value comes from the integral:

$$I = \int_0^\infty \int_0^{2\pi} \int_0^\pi \left[\int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{64}{\pi^2 a^6} e^{-4(r_1+r_2)/a} \frac{1}{[r_1^2+r_2^2-2r_1r_2\cos\theta_2]^{1/2}} r_1^2 \sin\theta_1 \, r_2^2 \sin\theta_2 \, d\theta_2 \, d\phi_2 \, dr_2 \right] d\theta_1 \, d\phi_1 \, dr_1$$

$$= \frac{64}{\pi^2 a^6}(2\pi)(4\pi) \int_0^\infty \int_0^\infty e^{-4(r_1+r_2)/a} r_1^2 r_2^2 \left[\int_0^\pi \frac{\sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}} d\theta_2 \right] dr_1 \, dr_2$$

Note that $\frac{d}{d\theta}[A + B\cos\theta]^{1/2} = \frac{-\frac{1}{2} B \sin\theta}{[A+B\cos\theta]^{1/2}}$, & for $A = r_1^2 + r_2^2$, $B = -2r_1 r_2$

so that $\int_0^\pi \frac{\sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}} d\theta_2 = \left[r_1^2+r_2^2-2r_1r_2\cos\theta_2\right]^{1/2}\Big|_{\theta_2=0}^{\pi} \cdot \left(\frac{-2}{-2r_1r_2}\right) = \frac{1}{r_1 r_2}\left[\sqrt{(r_1+r_2)^2} - \sqrt{(r_1-r_2)^2}\right]$

$= \frac{1}{r_1 r_2}\left[r_1 + r_2 - \sqrt{(r_1-r_2)^2}\right]$.

& $\sqrt{(r_1-r_2)^2} = r_1 - r_2$ if $r_2 < r_1$, $r_2 - r_1$ if $r_2 > r_1$, so the integral is:

$$I = \frac{64\cdot 8}{a^6}\left[\int_0^\infty \left[\int_0^{r_1} e^{-4(r_1+r_2)/a} r_1 r_2 (2r_2)\, dr_2 + \int_{r_1}^\infty e^{-4(r_1+r_2)/a} r_1 r_2 (2r_1)\, dr_2\right] dr_1\right]$$

$$= \frac{2\cdot 64\cdot 8}{a^6}\int_0^\infty e^{-4r_1/a} r_1 \left[\underbrace{\int_0^{r_1} e^{-4r_2/a} r_2^2\, dr_2}_{= \frac{1}{32}a^3\left(1 - e^{-4r_1/a}\left(1 + \frac{4r_1}{a} + \frac{8r_1^2}{a^2}\right)\right)} + \int_{r_1}^\infty e^{-4r_2/a} r_2 r_1\, dr_2\right] dr_1$$

$= \frac{r_1}{16} a^2 e^{-4r_1/a}\left(1 + \frac{4r_1}{a}\right)$

$$= \frac{16\cdot 64}{a^6}\int_0^\infty e^{-4r_1/a} \cdot \frac{a^3 r_1}{32}\left(1 - e^{-4r_1/a}(1 + 2r_1/a)\right) dr_1$$

$$= \frac{16\cdot 64}{32 a^3}\cdot\frac{5a^2}{128} = \boxed{\frac{5}{4a}}$$

b. $\left\langle \frac{e^2}{4\pi\epsilon_0 r_{12}} \right\rangle = \frac{(1.6\times 10^{-19} C)^2}{4\pi(8.85\times 10^{-12} C^2/Nm^2)} \cdot \frac{5}{4(.5\times 10^{-10} m)} \approx 5.8\times 10^{-18} J$

& $1J = 6.24\times 10^{18} eV$, so $\left\langle \frac{e^2}{4\pi\epsilon_0 r_{12}} \right\rangle \approx 5.8\times 10^{-18} J \cdot \frac{6.24\times 10^{18} eV}{J} \approx 36.2 eV$

so our refinement gives:

$4\cdot 2\cdot(-13.6 eV) + 36.2 eV = \boxed{-73. eV}$, (closer to $-79 eV$.)

Problem 32.2

For spin-1 bosons, the $|00\rangle$ state is:

$$|00\rangle = \frac{1}{\sqrt{3}}|11\rangle|1{-1}\rangle - \frac{1}{\sqrt{3}}|10\rangle|10\rangle + \frac{1}{\sqrt{3}}|1{-1}\rangle|11\rangle.$$

so $P|00\rangle = \frac{1}{\sqrt{3}}|1{-1}\rangle|11\rangle - \frac{1}{\sqrt{3}}|10\rangle|10\rangle + \frac{1}{\sqrt{3}}|11\rangle|1{-1}\rangle = |00\rangle$

i.e. the spin state is symmetric under particle exchange.

For the spatial part of the wave function, we have:

$$\psi_\pm(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_p(x_1)\psi_q(x_2) \pm \psi_p(x_2)\psi_q(x_1)\right].$$

we want the symmetric part; so

$$\psi(x_1,x_2) = \boxed{\psi_+|00\rangle = \sqrt{\frac{2}{3}}\frac{1}{a}\left[\sin\left(\frac{p\pi x_1}{a}\right)\sin\left(\frac{q\pi x_2}{a}\right) + \sin\left(\frac{p\pi x_2}{a}\right)\sin\left(\frac{q\pi x_1}{a}\right)\right]\left(|11\rangle|1{-1}\rangle - |10\rangle|10\rangle + |1{-1}\rangle|11\rangle\right)}$$

Problem 32.3

The ground state for 2 identical fermions (spin ½ in this case) in a harmonic oscillator is:

$$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\right]|10\rangle$$

since $P|10\rangle = |10\rangle$ we must take the antisymmetric spatial combination, so the full expression is:

$$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}\sqrt{\frac{2m\omega}{\hbar}}\left[x_2 - x_1\right]|10\rangle.$$

$$= \boxed{\frac{1}{\sqrt{2\pi}}\frac{m\omega}{\hbar}e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}\left[x_2-x_1\right]\left[|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\tfrac{-1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right]}$$

The energy of this state will be: $E = E_0 + E_1 = \frac{1}{2}\hbar\omega + \hbar\omega = \frac{3}{2}\hbar\omega$

(& not $\hbar\omega$ as would be the case for the symmetric spatial wavefunction).