Problem 31.1 (Griffiths 5.1)

Define: $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$, $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$

a. we have $\vec{r}_2 = \vec{r}_1 - \vec{r}$, so $\vec{R} = \frac{1}{m_1+m_2}\left[m_1\vec{r}_1 + m_2\vec{r}_1 - m_2\vec{r}\right] = \vec{r}_1 - \frac{\mu}{m_1}\vec{r} \Rightarrow \boxed{\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r}}$

& $\vec{r}_1 = \vec{r} + \vec{r}_2$ so $\vec{R} = \frac{1}{m_1+m_2}\left[m_1\vec{r} + m_1\vec{r}_2 + m_2\vec{r}_2\right] = \vec{r}_2 + \frac{\mu}{m_2}\vec{r} \Rightarrow \boxed{\vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r}}$

To find ∇_1, think of a function $f(\vec{r},\vec{R})$, then using the chain rule:

$$\frac{\partial}{\partial x_1} f(\vec{r},\vec{R}) = \nabla_r f \cdot \frac{\partial \vec{r}}{\partial x_1} + \nabla_R f \cdot \frac{\partial \vec{R}}{\partial x_1}$$

& $\frac{\partial \vec{r}}{\partial x_1} = \hat{x}$, $\frac{\partial \vec{R}}{\partial x_1} = \frac{\mu}{m_2}\hat{x}$, so

$$= \left[\nabla_r f + \frac{\mu}{m_2}\nabla_R f\right]\cdot\hat{x}$$

& sim: $\frac{\partial f}{\partial y_1} = \left[\nabla_r f + \frac{\mu}{m_2}\nabla_R f\right]\cdot\hat{y}$, $\frac{\partial f}{\partial z_1} = \left[\nabla_r f + \frac{\mu}{m_2}\nabla_R f\right]\cdot\hat{z}$

so $\nabla_1 f = \left[\nabla_r + \frac{\mu}{m_2}\nabla_R\right] f$ & we can write: $\boxed{\nabla_1 \longrightarrow \nabla_r + \frac{\mu}{m_2}\nabla_R.}$

For ∇_2: $\frac{\partial}{\partial x_2} f(\vec{r},\vec{R}) = \nabla_r f \cdot \frac{\partial \vec{r}}{\partial x_2} + \nabla_R f \cdot \frac{\partial \vec{R}}{\partial x_2} = \nabla_r f(-\hat{x}) + \nabla_R f\cdot\left(\frac{\mu}{m_1}\hat{x}\right)$

$$= \left[-\nabla_r f + \frac{\mu}{m_1}\nabla_R f\right]\cdot\hat{x}$$

& as above, this gives: $\boxed{\nabla_2 \longrightarrow -\nabla_r + \frac{\mu}{m_1}\nabla_R}$

b. Starting from: $\left[\frac{-\hbar^2}{2m_1}\nabla_1^2 + \frac{-\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r})\right]\Psi = E\Psi$, we just insert our substitutions from above:

$$-\frac{\hbar^2}{2}\left[\frac{1}{m_1}\left(\nabla_r^2 + 2\frac{\mu}{m_2}\nabla_r\cdot\nabla_R + \frac{\mu^2}{m_2^2}\nabla_R^2\right) + \frac{1}{m_2}\left(\nabla_r^2 - 2\frac{\mu}{m_1}\nabla_R\cdot\nabla_r + \frac{\mu^2}{m_1^2}\nabla_R^2\right)\right]\Psi + V\Psi = E\Psi$$

$$= -\frac{\hbar^2}{2}\left[\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla_r^2 + \left(\frac{\mu^2}{m_1 m_2^2} + \frac{\mu^2}{m_2 m_1^2}\right)\nabla_R^2\right]\Psi + V\Psi = E\Psi$$

& $\frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2} = \frac{1}{\mu}$, $\frac{\mu^2}{m_1 m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{\mu}{m_1 m_2} = \frac{1}{m_1+m_2}$

$$= \boxed{-\frac{\hbar^2}{2}\left[\frac{1}{\mu}\nabla_r^2 + \frac{1}{m_1+m_2}\nabla_R^2\right]\Psi(\vec{r},\vec{R}) + V(\vec{r})\Psi(\vec{r},\vec{R}) = E\Psi(\vec{r},\vec{R})}$$ as desired

c. Let $\Psi(\vec{r},\vec{R}) = \psi_R(\vec{R})\,\psi_r(\vec{r})$ — inserting in gives:

$$\underbrace{\frac{1}{\psi_r}\left[\frac{-\hbar^2}{2}\frac{1}{\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_r\right]}_{=E_r} + \underbrace{\frac{1}{\psi_R}\left[\frac{-\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R\right]}_{E_R} = E$$

now we have 2 eqns:

$\boxed{-\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_r = E_r\psi_r}$ & $\boxed{-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R = E_R\psi_R}$

for an effective particle of mass μ under the influence of $V(\vec{r})$; for a free particle of mass m_1+m_2.

the total energy is $E = E_r + E_R$.

Problem 31.2 (Griffiths 5.6)

a. For distinguishable particles, $\psi(x_1,x_2) = \psi_n(x_1)\psi_l(x_2) = \frac{2}{a}\sin\left(\frac{n\pi x_1}{a}\right)\sin\left(\frac{l\pi x_2}{a}\right)$
so we compute:

$$\langle (x_1-x_2)^2\rangle = \int_0^a\int_0^a \left(\frac{2}{a}\right)^2 \sin^2\left(\frac{n\pi x_1}{a}\right)(x_1^2 - 2x_1x_2 + x_2^2)\sin^2\left(\frac{l\pi x_2}{a}\right) dx_1\, dx_2$$

$$= \left(\frac{2}{a}\right)^2\left[\int_0^a x_1^2\sin^2\left(\frac{n\pi x_1}{a}\right)dx_1 + \int_0^a x_2^2\sin^2\left(\frac{l\pi x_2}{a}\right)dx_2 - \int_0^a\int_0^a 2x_1x_2\sin\left(\frac{n\pi x_1}{a}\right)\sin\left(\frac{l\pi x_2}{a}\right)dx_1dx_2\right]$$

$$= \left[\langle x_1^2\rangle + \langle x_2^2\rangle - 2\langle x_1\rangle\langle x_2\rangle\right]$$

so we know $\langle x_1^2\rangle = \frac{1}{6}a^2\left(2-\frac{3}{n^2\pi^2}\right)$ $\langle x_2^2\rangle = \frac{1}{6}a^2\left(2-\frac{3}{l^2\pi^2}\right)$
so

$\langle x_1\rangle = a/2$, $\langle x_2\rangle = a/2$, so

$$\langle (x_1-x_2)^2\rangle = \left[\frac{2}{3}a^2 - \frac{a^2}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right)\right] - \frac{a^2}{2} = \boxed{a^2\left[\frac{1}{6} - \frac{a^2}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right)\right]}$$

b. For bosons (fermions): $\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_n(x_1)\psi_l(x_2) \pm \psi_n(x_2)\psi_l(x_1)\right]$
so

$$\langle (x_1-x_2)^2\rangle = \frac{1}{2}\int_0^a\int_0^a\Big[\psi_n(x_1)^2\psi_l^2(x_2)(x_1-x_2)^2 + \psi_n(x_2)^2\psi_l(x_1)^2(x_1-x_2)^2$$
$$\pm 2\psi_n(x_1)\psi_n(x_2)\psi_l(x_1)\psi_l(x_2)(x_1-x_2)^2\Big]dx_1\,dx_2.$$

$$= \left[\langle x_1^2\rangle + \langle x_2^2\rangle - 2\langle x_1\rangle\langle x_2\rangle \pm \int_0^a\int_0^a (-2x_1x_2)\psi_n(x_1)\psi_l(x_2)\psi_n(x_2)\psi_l(x_1)\,dx_1dx_2\right]$$

(where terms like: $\int_0^a\int_0^a \psi_n(x_1)\psi_l(x_2)\psi_n(x_2)\psi_l(x_1)x_1^2\,dx_1dx_2$
$$= \int_0^a \psi_n(x_1)\psi_l(x_1)x_1^2\,dx_1 \underbrace{\int_0^a \psi_n(x_2)\psi_l(x_2)\,dx_2}_{=\emptyset \text{ by orthogonality}} = \emptyset$$

we need: $\int_0^a \psi_n(x_1)x_1\psi_l(x_1)\,dx_1\int_0^a\psi_n(x_2)x_2\psi_l(x_2)\,dx_2$
$$= \left[\frac{2}{a}\int_0^a \psi_n(x_1)x_1\psi_l(x_1)\,dx_1\right]^2$$
$$= \frac{4anl(-1+(-1)^{n+l})}{(l^2-n^2)^2\pi^2}$$

so $$\langle (x_1-x_2)^2\rangle = \frac{2}{3}a^2 - \frac{a^2}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right) - \frac{a^2}{2} \mp 2\left[\frac{4anl(-1+(-1)^{n+l})}{(l^2-n^2)^2\pi^2}\right]^2$$

for ψ_+ (bosons)

$$= \boxed{a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right)\right] \mp \frac{32n^2l^2a^2}{(l^2-n^2)^4\pi^4}\left(-1+(-1)^{n+l}\right)^2}$$

for ψ_- (fermions)