Problem 30.1

The top state has $|l\,m_l\rangle|s_e\,m_s\rangle$ w/ $l=1$, $s_e=\frac{1}{2}$, $m_l=1$, $m_s=\frac{1}{2}$, the $m_l+m_s=\frac{3}{2}\equiv m$ & this ~~must~~ be a $j=\frac{3}{2}$ state, so:

$$\boxed{|\underbrace{\tfrac{3}{2}}_{j}\,\underbrace{\tfrac{3}{2}}_{m}\rangle = |1\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle}$$

acting w/ $J_- = L_- + S_-$, we have:

$$J_-|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2}-\tfrac{3}{2}\cdot\tfrac{1}{2}}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle \qquad (1)$$

and

$$(L_-+S_-)|1\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{2-\cancel{0}}\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2}-\tfrac{1}{2}(-\tfrac{1}{2})}\,|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$= \sqrt{2}\,\hbar\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle. \qquad (2)$$

putting (1) & (2) together,

$$J_-|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = (L_-+S_-)|1\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$\sqrt{3}\,\hbar\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\,|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\,\hbar\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

we get

$$\boxed{|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}\left[|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]}$$

Now acting on this state w/ J_-:

$$J_-|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2}-\tfrac{1}{2}(-\tfrac{1}{2})}\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = 2\hbar\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle. \qquad (3)$$

&

$$(L_-+S_-)|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \hbar\sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \quad (\text{since } S_-|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = 0).$$

$$(L_-+S_-)|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2}-\tfrac{1}{2}(-\tfrac{1}{2})}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle.$$

then:

$$(L_-+S_-)\tfrac{1}{\sqrt{3}}\left[|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right] = \tfrac{1}{\sqrt{3}}\left[\sqrt{2}\,\hbar\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\left(\sqrt{2}\,\hbar\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)\right]$$

$$= \sqrt{\tfrac{2}{3}}\,\hbar\left[2\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right] \qquad (4).$$

putting (3) & (4) together:

$$\boxed{|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}\left[2\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]} = \tfrac{1}{\sqrt{3}}\left[\sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + |1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$$

and we know the bottom state is:

$$\boxed{|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = |1\,-1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

Problem 30.1 (continued)

To get the $j = 1/2$ series, we start w/ a general linear combination of $m = 1/2$:

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = a\,|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + b\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

and demand that: $\langle \tfrac{3}{2}\,\tfrac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \emptyset$

$$= \frac{1}{\sqrt{3}}\left[\langle 1\,1|\langle \tfrac{1}{2}\,{-\tfrac{1}{2}}| + \sqrt{2}\,\langle 1\,0|\langle \tfrac{1}{2}\,\tfrac{1}{2}|\right]\left[a\,|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + b\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$$

$$= \frac{1}{\sqrt{3}}\left[a\langle 1\,1|1\,1\rangle\langle \tfrac{1}{2}\,{-\tfrac{1}{2}}|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + b\underbrace{\langle 1\,1|1\,0\rangle\langle \tfrac{1}{2}\,{-\tfrac{1}{2}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle}_{\emptyset} + \sqrt{2}\,a\underbrace{\langle 1\,0|1\,1\rangle\langle \tfrac{1}{2}\,\tfrac{1}{2}|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle}_{\emptyset} + \sqrt{2}\,b\langle 1\,0|1\,0\rangle\langle \tfrac{1}{2}\,\tfrac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$$

$$= \frac{1}{\sqrt{3}}\left[a + \sqrt{2}\,b\right] = \emptyset \Rightarrow a = -\sqrt{2}\,b.$$

& to normalize it, we have: $a^2 + b^2 = 1 \rightarrow 2b^2 + b^2 = 1 \rightarrow b = 1/\sqrt{3}$

then

$$\boxed{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left[|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - \sqrt{2}\,|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right].}$$

& this state clearly has $m = 1/2$, & is $\perp$ to $|\tfrac{3}{2}\,\tfrac{1}{2}\rangle$ - it must be (here) $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$.

Applying the lowering operator gives:

$$J_-\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}(-\tfrac{1}{2})}\;|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = \hbar\,|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle. \qquad (5)$$

&

$$(L_- + S_-)\frac{1}{\sqrt{3}}\left[|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - \sqrt{2}\,|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right] = \frac{1}{\sqrt{3}}\left[\hbar\sqrt{2}\,|1\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|1\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{2}\,\hbar\sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right]$$

$$= \frac{\hbar}{\sqrt{3}}\left[\sqrt{2}\,|1\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - |1\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right] \qquad (6)$$

& putting (5) & (6) together:

$$\boxed{|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|1\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - |1\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right]}$$

From the table, we have:

$|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = |1\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ ✓

$|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left[|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$ ✓

$|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + |1\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$ ✓

$|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle = |1\,{-1}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$ ✓

& $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\,|1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - |1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$ ✓

$|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{3}}\left[|1\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - \sqrt{2}\,|1\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$ ✓

Problem 10.2 (Griffiths 4.35)

a. 2 quarks can have spin $\frac{1}{2}+\frac{1}{2}=1$ or $\frac{1}{2}-\frac{1}{2}=\emptyset$, so we must add a particle w/ $s=\emptyset$ or 1 to a 3rd quark ($s=1/2$), giving:

$$0+1/2=\frac{1}{2}$$

$$1+1/2=\boxed{3/2}$$
$$1-1/2=\boxed{1/2}$$

so $s=1/2$ or $3/2$.

b. 2 quarks can have spin $\boxed{1 \text{ or } \emptyset.}$