Problem 29.1 (Griffiths 4.27)

a. For $\chi = A\begin{pmatrix}3i\\4\end{pmatrix}$, $\chi^\dagger\chi = A^2(+9+16) = A^2 25 = 1 \rightarrow \boxed{A = 1/5}$

b. $\langle S_x\rangle = \left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3i/5\\4/5\end{pmatrix} = \frac{\hbar}{2}\left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\begin{pmatrix}4/5\\3i/5\end{pmatrix} = \boxed{\emptyset}$

$\langle S_y\rangle = \left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}3i/5\\4/5\end{pmatrix} = \frac{\hbar}{2}\left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\begin{pmatrix}-4i/5\\-3/5\end{pmatrix} = \frac{\hbar}{2}\left(-\frac{12}{25}-\frac{12}{25}\right) = \boxed{\frac{-12\hbar}{25}}$

$\langle S_z\rangle = \left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3i/5\\4/5\end{pmatrix} = \frac{\hbar}{2}\left(-\frac{3i}{5}\ \ \frac{4}{5}\right)\begin{pmatrix}3i/5\\-4/5\end{pmatrix} = \frac{\hbar}{2}\left(\frac{9}{25}-\frac{16}{25}\right) = \boxed{\frac{-7\hbar}{50}}$

c. We need $S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

so $\langle S_x^2\rangle = \frac{\hbar^2}{4} = \langle S_y^2\rangle = \langle S_z^2\rangle \rightarrow \sigma_{S_x} = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \emptyset} = \boxed{\hbar/2}$

$\sigma_{S_y} = \sqrt{\frac{\hbar^2}{4} - \frac{144\hbar^2}{625}} = \sqrt{\frac{49\hbar^2}{2500}} = \boxed{\frac{7\hbar}{50}}$

$\sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \frac{49\hbar^2}{2500}} = \sqrt{\frac{144\hbar^2}{625}} = \boxed{\frac{12\hbar}{25}}$

d. The uncertainty relations are: $\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle S_z\rangle|$

check $\frac{\hbar}{2}\frac{7\hbar}{50} = \frac{7\hbar^2}{100} \overset{?}{\geq} \frac{\hbar}{2}\cdot\frac{7\hbar}{50} = \frac{7\hbar^2}{100}$ ✓

$\sigma_{S_z}\sigma_{S_x} \geq \frac{\hbar}{2}|\langle S_y\rangle|$

check $\frac{12\hbar}{25}\cdot\frac{\hbar}{2} = \frac{6\hbar^2}{25} \overset{?}{\geq} \frac{\hbar}{2}\frac{12\hbar}{25} = \frac{6\hbar^2}{25}$ ✓

$\sigma_{S_y}\sigma_{S_z} \geq \frac{\hbar}{2}|\langle S_x\rangle|$

$\frac{7\hbar}{50}\frac{12\hbar}{25} = \frac{42\hbar^2}{625} \overset{?}{\geq} \frac{\hbar}{2}\cdot\emptyset = \emptyset$ ✓

Problem 29.2

We are given $\chi = \chi_+$, the up state for the z-component of spin. To find the probability associated w/ an S_y measurement, we need the decomposition of χ_+ in the eigenvectors of S_y, χ_+^y, χ_-^y.

From $S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ we can see that $\chi_+^y \sim \begin{pmatrix}1\\i\end{pmatrix}$, since $S_y\chi_+^y \doteq \frac{\hbar}{2}\begin{pmatrix}1\\i\end{pmatrix}$

& $\chi_-^y \sim \begin{pmatrix}i\\1\end{pmatrix}$: $S_y\chi_-^y = -\frac{\hbar}{2}\begin{pmatrix}i\\1\end{pmatrix}$

we need to normalize: $\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, $\chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}$ – to write $\chi_+ \doteq \begin{pmatrix}1\\0\end{pmatrix}$ in this basis, we need $\chi_+^\dagger\chi_+^y = \frac{1}{\sqrt{2}}$, $\underbrace{\chi_+^\dagger\chi_-^y = \frac{1}{\sqrt{2}}i}_{\text{so } \chi_-^{y\dagger}\chi_+ = -\frac{1}{\sqrt{2}}i}$, then:

$$\chi_+ = \frac{1}{\sqrt{2}}\chi_+^y - \frac{i}{\sqrt{2}}\chi_-^y$$

$\rightarrow$ the probability of measuring $\frac{\hbar}{2}$ for the x-component of spin is $\left(\frac{1}{\sqrt{2}}\right)^2 = 1/2$.

Problem 29.3

a. Starting w/ $\chi(t) = A e^{-iE_+ t/\hbar}$ w/ $E_+ = -\frac{\hbar\gamma}{2}(B_0 + \alpha z)$.

Then: $\frac{\partial}{\partial z}\chi(t) = \frac{-i}{\hbar}\left(-\frac{\hbar\gamma}{2}\alpha t\right) A e^{-iE_+ t/\hbar} = \frac{i\alpha\gamma}{2} t \chi$

$$\frac{\partial^2}{\partial z^2}\chi(t) = \left(\frac{i\alpha\gamma}{2} t\right)^2 \chi \quad \& \quad \frac{\partial}{\partial t}\chi = -\frac{i}{\hbar}\left(-\frac{\hbar\gamma}{2}(B_0 + \alpha z)\right)\chi = \frac{i\gamma}{2}(B_0 + \alpha z)\chi$$

& Schrödinger's eqn reads:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \chi}{\partial z^2} + \left(-\gamma(B_0 + \alpha z) S_z\right)\chi \overset{?}{=} i\hbar \frac{\partial \chi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\left(-\frac{\alpha^2\gamma^2}{4} t^2\right)\chi \underbrace{- \gamma(B_0 + \alpha z)\frac{\hbar}{2}\chi = i\hbar\left(\frac{i\gamma}{2}(B_0 + \alpha z)\right)\chi}_{\text{cancel}}$$

So we have: $\frac{\hbar^2 \alpha^2 \gamma^2 t^2}{8m}\chi \overset{?}{=} \emptyset$

not true, but provide α is "small", approximately correct: $O(\alpha^2) = \emptyset$.

b. For $H = -\gamma \vec{S}\cdot\vec{B}$ w/ $\vec{B} = -\alpha x \hat{x} + (B_0 + \alpha z)\hat{z}$, we have:

$$H = -\gamma\left[-\alpha x S_x + (B_0 + \alpha z) S_z\right].$$

& we want the energies associated w/: $H\chi = E\chi$:

$$H = \alpha\gamma x \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \gamma(B_0 + \alpha z)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\doteq \frac{\hbar\gamma}{2}\begin{pmatrix} -(B_0 + \alpha z) & \alpha x \\ \alpha x & +(B_0 + \alpha z) \end{pmatrix}$$

for a matrix of the form: $\mathbb{A} \doteq \begin{pmatrix} -a & b \\ b & +a \end{pmatrix}$, the e-vals are:

$$(a - \lambda)(-a - \lambda) - b^2 = \emptyset \Rightarrow -a^2 + \lambda^2 - b^2 = \emptyset$$

so that

$$\lambda = \pm\sqrt{b^2 + a^2}$$

for our matrix, $a = \frac{\hbar\gamma}{2}(B_0 + \alpha z)$, $b = \frac{\hbar}{2}\alpha\gamma x$.

$$\lambda_\pm = \pm\frac{\hbar\gamma}{2}\sqrt{(B_0 + \alpha z)^2 + (\alpha x)^2} = \pm\frac{\hbar\gamma}{2}(B_0 + \alpha z)\sqrt{1 + \left(\frac{\alpha x}{B_0 + \alpha z}\right)^2}$$

the energies are approximately (using $\frac{\alpha x}{B_0 + \alpha z} \approx \epsilon$ small, & $\sqrt{\rho + \epsilon} \approx (\rho + \frac{1}{2}\epsilon)$)

$$E_\mp \approx \pm\frac{\hbar\gamma}{2}(B_0 + \alpha z)\left(1 + \frac{1}{2}\left(\frac{\alpha x}{B_0 + \alpha z}\right)^2\right)$$

$$\approx \pm\frac{\hbar\gamma}{2}(B_0 + \alpha z) + O\left(\left(\frac{\alpha x}{B_0 + \alpha z}\right)^2\right)$$