Problem 28.1 (Griffiths 4.31)

For spin 1: $\chi_+ \doteq \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\chi_0 \doteq \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\chi_- \doteq \begin{pmatrix}0\\0\\1\end{pmatrix}$.

& (4.136) says: $S_\pm|s\,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm1)}\,|s\,(m\pm1)\rangle$.
we also know that
$S_z|s\,m\rangle = \hbar m|s\,m\rangle$ & $S^2|s\,m\rangle = \hbar^2 s(s+1)|s\,m\rangle$.

Take S_z to have the form $S_z \doteq \begin{pmatrix}a & b & c\\ b^* & d & e\\ c^* & e^* & f\end{pmatrix}$ for complex $\{a,b,c,d,e,f\}$. ← assume Hermitian.

Then $S_z\cdot\chi_+ \doteq \begin{pmatrix}a\\b^*\\c^*\end{pmatrix} = \hbar\cdot1\cdot\chi_+ \doteq \begin{pmatrix}\hbar\\0\\0\end{pmatrix}$ so $a=\hbar$, $b=c=\emptyset$.

$S_z\chi_0 \doteq \begin{pmatrix}b\\d\\e^*\end{pmatrix} = \hbar\cdot\emptyset\cdot\chi_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$ so $b=d=e=\emptyset$

$S_z\chi_- \doteq \begin{pmatrix}c\\e\\f\end{pmatrix} = -\hbar\cdot1\cdot\chi_- = \begin{pmatrix}0\\0\\-\hbar\end{pmatrix}$ so $f=-\hbar$

and we have: $\boxed{S_z \doteq \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}}$.

Now take $S_+ \doteq \begin{pmatrix}a&b&c\\d&e&f\\g&j&k\end{pmatrix}$ & note that $S_+\chi_+ = \emptyset$, so:

$S_+\chi_+ \doteq \begin{pmatrix}a\\d\\g\end{pmatrix} = \emptyset \Rightarrow a=d=g=\emptyset$. $S_+\chi_0 = \hbar\sqrt{2-\emptyset}\,\chi_+$
& that gives $S_+\chi_0 = \begin{pmatrix}b\\e\\j\end{pmatrix} = \hbar\sqrt2\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow b=\sqrt2\hbar$, $e=j=\emptyset$. $S_+\chi_- = \hbar\sqrt{2-\emptyset}\,\chi_0$
& tells us: $S_+\chi_- = \begin{pmatrix}c\\f\\k\end{pmatrix} = \hbar\sqrt2\begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow f=\sqrt2\hbar$, $c=k=\emptyset$.

The S_+ matrix is: $S_+ \doteq \sqrt2\,\hbar\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

Sim.: $S_-\chi_- = \emptyset$, $S_-\chi_0 = \hbar\sqrt{2-\emptyset}\,\chi_-$, $S_-\chi_+ = \hbar\sqrt{2-\emptyset}\,\chi_0$
gives (from the same starting point as S_+):

$\begin{pmatrix}c\\f\\k\end{pmatrix} = \sqrt2\hbar\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}b\\e\\j\end{pmatrix} = \sqrt2\hbar\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\begin{pmatrix}a\\d\\g\end{pmatrix} = \sqrt2\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$

then $c=f=k=b=e=a=g=\emptyset$, & $j=d=\sqrt2\hbar$, so $S_- \doteq \sqrt2\hbar\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

Inverting the defining relationship: $S_+ = S_x + iS_y$
$S_- = S_x - iS_y$

we find: $S_x = \frac{1}{2}(S_+ + S_-)$ & $S_y = \frac{1}{2i}(S_+ - S_-)$.

so $\boxed{S_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$ & $\boxed{S_y = -\frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}$

Problem 28.2 (Griffiths 4.33)

a. $H = -\vec{m}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\gamma S_z B_0\cos(\omega t)$.
using the matrix form of S_z, we have:

$$\boxed{H \doteq -\gamma B_0 \frac{\hbar}{2}\cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

b. Our unknown function is $\chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, & Schrödinger's eqn. tells us:

$$H\chi(t) = i\hbar\frac{\partial\chi}{\partial t}$$

$$-\gamma B_0\frac{\hbar}{2}\cos(\omega t)\begin{pmatrix} a(t) \\ -b(t) \end{pmatrix} = i\hbar\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix}$$

we have 2 eqn.s: $\dot{a}(t) = \frac{i\gamma B_0}{2}\cos(\omega t)\,a(t)$ & $\dot{b}(t) = \frac{-i\gamma B_0}{2}\cos(\omega t)\,b(t)$

For the a(t) eqn. we can write:

$$\frac{d}{dt}\log(a(t)) = +\frac{1}{\omega}\frac{d}{dt}\left(\frac{i\gamma B_0}{2}\sin(\omega t)\right)$$

so $a(t) = a_0 e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)}$ & sim. $b(t) = b_0 e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)}$

at $t = \emptyset$, we are given $\chi(\emptyset) = \chi_+^{(x)}$ & from $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we know that the $\chi_+^{(x)}$ state is represented by the vector:

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so } a_0 = \frac{1}{\sqrt{2}} = b_0$$

& $\boxed{\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix}}$

c. We want: $\chi_-^{(x)}\cdot\chi(t) = \frac{1}{\sqrt{2}}(1 \quad -1)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-\frac{i\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix}$

$$= \frac{1}{2}\left[2i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)\right]$$

then the probability is: $\boxed{|\chi_-^{(x)}\chi(t)|^2 = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)}$

d. To get this expression $= 1$, we must have: $\frac{\gamma B_0}{2\omega}\sin(\omega t) = \pi/2$, or

$$\frac{\gamma B_0}{2\omega} = \pi/2 \Rightarrow \boxed{\frac{\omega\pi}{\gamma} = B_0}$$

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