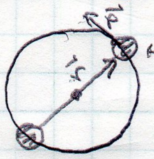
Problem 27.1 (Griffiths 4.24)

a. The classical angular momentum is:



$$\vec{L}_1 = \vec{r}\times\vec{p} = \underbrace{\frac{a}{2}}_{r}\cdot\underbrace{m\omega\left(\frac{a}{2}\right)}_{p}\sin\left(\frac{\pi}{2}\right)\hat{z} = \frac{m\omega a^2}{4}\hat{z}$$

There are two particles, so $\vec{L} = \vec{L}_1 + \vec{L}_2 = \frac{m\omega a^2}{2}\hat{z}$

Then the magnitude is $L^2 = \frac{m^2\omega^2 a^4}{4}$

The energy of the system is: $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} = \frac{1}{m}\left(m\omega \frac{a}{2}\right)^2 = \frac{m\omega^2 a^2}{4}$

so that

$$H = \frac{1}{ma^2}L^2$$

The eigenfunctions of L^2 are the $Y_\ell^m(\theta,\phi)$, we know that

$$L^2 Y_\ell^m = \hbar^2 \ell(\ell+1)$$

& since $H \sim L^2$, these are the energy eigenfunctions, too.

The eigenvalue is the energy, since Schrödinger's eqn. reads:

$$H\Psi = E\Psi$$

if we take $\Psi = Y_\ell^m(\theta,\phi)$, we have:

$$H\Psi = \frac{1}{ma^2}L^2 Y_\ell^m = \frac{\hbar^2}{ma^2}\ell(\ell+1)\, Y_\ell^m = E_\ell \Psi$$

and then: $\boxed{E_\ell = \frac{\hbar^2}{\underbrace{m}_{\text{mass}}a^2}\ell(\ell+1) \quad \text{for } \ell = 0, 1, 2, \dots}$

b. The normalized eigenfunctions are $\boxed{\Psi_{\ell m}(\theta,\phi) = Y_\ell^m(\theta,\phi)}$ and there are $2\ell+1$ values of m for each ℓ, so the degeneracy is $\boxed{2\ell+1}$

Problem 27.2

For $s = 1/2$, $\ell = 1$, we have $\vec{J} = \vec{L} + \vec{S}$, & the z-component is:

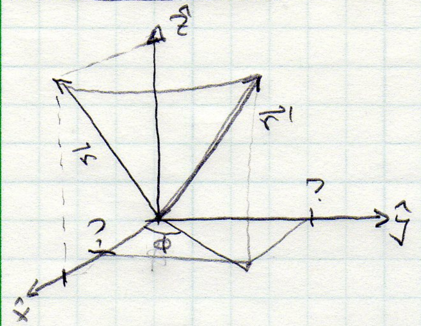
$$J_z = L_z + S_z$$

The orbital z-component can take values $L_z = -1, 0, 1$ (since $\ell = 1$ & $-1 \le m \le 1$) & the spin z-component has $S_z = -1/2, 1/2$, so the sums are:

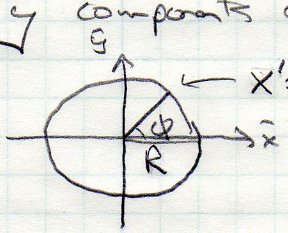
J_z =	L_z +	S_z
$-3/2$	-1	$-1/2$
$-1/2$	-1	$1/2$
$-1/2$	0	$-1/2$
$1/2$	0	$1/2$
$1/2$	1	$-1/2$
$3/2$	1	$1/2$

so we can get z-components: $\boxed{-3/2, -1/2, 1/2, 3/2}$

Problem 27.3



For $\vec{r} = R\hat{x} + z\hat{z}$, we want to find $\vec{r}\,'$
the z-component is the same, $z' = z$
& the x & y components are:



$x' = R\cos\phi$

$y' = R\sin\phi$

so $\vec{r}\,' = R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y} + z\hat{z} \approx R\hat{x} + R\phi\hat{y} + z\hat{z}$

then

$$\vec{r}\,' - \vec{r} = R\phi\hat{y}$$

Meanwhile, for $\vec{\Omega} = \phi\hat{z}$, we have

$$\vec{\Omega} \times \vec{r} = \phi\hat{z} \times (R\hat{x} + z\hat{z}) = \phi R\,\hat{z}\times\hat{x} + \phi z\,\cancel{\hat{z}\times\hat{z}}$$

$$= \phi R\hat{y}$$

so it is true that $\boxed{\vec{r}\,' - \vec{r} = \vec{\Omega} \times \vec{r}.}$