Problem 26.1

a. $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$

then: $[L_z, x] = \cancel{[xp_y, x]} - [yp_x, x] = -(yp_x x - \underbrace{xyp_x}_{= yxp_x = y(i\hbar + p_x x)}) = +i\hbar y$

$[L_z, y] = [xp_y, y] - \cancel{[yp_x, y]} = xp_y y - \underbrace{yxp_y}_{= xyp_y = x(p_y y + i\hbar)} = -i\hbar x$

$[L_z, z] = \cancel{[xp_y, z]} - \cancel{[yp_x, z]} = \varnothing$

&

$[L_z, p_x] = [xp_y, p_x] - \cancel{[yp_x, p_x]} = \underbrace{xp_y p_x}_{= xp_x p_y = (p_x x + i\hbar) p_y} - p_x x p_y = i\hbar p_y$

$[L_z, p_y] = \cancel{[xp_y, p_y]} - [yp_x, p_y] = -(\underbrace{yp_x p_y}_{= yp_y p_x = (p_y y + i\hbar) p_x} - p_y y p_x) = -i\hbar p_x$

$[L_z, p_z] = \cancel{[xp_y, p_z]} - \cancel{[p_z, xp_y]} = \varnothing$

b. $[L_z, L_x] = [L_z, yp_z] - [L_z, zp_y]$

& $[L_z, yp_z] = [L_z, y] p_z = -i\hbar x p_z$

$[L_z, zp_y] = [L_z, p_y] z = -i\hbar z p_x$

so $[L_z, L_x] = i\hbar(zp_x - xp_z) = i\hbar L_y$.

c. $[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + \cancel{[L_z, z^2]}$

& $[L_z, x^2] = \underbrace{L_z x}_{= xL_z + i\hbar y} x - x \underbrace{x L_z}_{= L_z x - i\hbar y} = 2i\hbar yx$

$[L_z, y^2] = \underbrace{L_z y}_{= yL_z - i\hbar x} y - y \underbrace{y L_z}_{= L_z y - i\hbar x} = -2i\hbar xy$

so $[L_z, r^2] = 2i\hbar yx - 2i\hbar xy = \varnothing$

$[L_z, p^2] = [L_z, p_x^2] + [L_z, p_y^2] + \cancel{[L_z, p_z^2]}$ w/

$[L_z, p_x^2] = \underbrace{L_z p_x}_{= p_x L_z + i\hbar p_y} p_x - p_x \underbrace{p_x L_z}_{= (L_z p_x - i\hbar p_y)} = 2i\hbar p_y p_x$

&

$[L_z, p_y^2] = \underbrace{L_z p_y}_{p_y L_z - i\hbar p_x} p_y - p_y \underbrace{p_y L_z}_{= L_z p_y + i\hbar p_x} = -2i\hbar p_x p_y$

so $[L_z, p^2] = 2i\hbar p_y p_x - 2i\hbar p_x p_y = \varnothing$.

d. For the Hamiltonian: $[H, L_z] = [\frac{p^2}{2m}, L_z] + [V(r), L_z]$

the first term, depending on p^2, is zero as established in part c: $[p^2, L_z] = \emptyset$, & that is the same result we would get for L_x or L_y, i.e. $[p^2, \vec{L}] = \emptyset$.

To mix it up, we'll do the potential commutator w/ a test function $g(\vec{r})$; using $\vec{r}\times\vec{p} \longrightarrow \vec{r}\times\frac{\hbar}{i}\nabla$

$$[V(r), \vec{L}]g(\vec{r}) = V(r)\vec{r}\times\frac{\hbar}{i}\nabla g - \vec{r}\times\frac{\hbar}{i}\nabla(V(r)g)$$

$$= \frac{\hbar}{i}[\cancel{V(r)\,\vec{r}\times\nabla g} - \cancel{V(r)\,\vec{r}\times\nabla g} - g(\vec{r})\,\vec{r}\times\nabla V]$$

$$= i\hbar g(\vec{r})\,\vec{r}\times\nabla V$$

for a potential depending only on r, $\nabla V = \frac{dV}{dr}\nabla r$ & $\nabla r = \hat{r}$, so

$$= i\hbar g(\vec{r})\,\vec{r}\times(\frac{dV}{dr}\hat{r})$$

$$= \emptyset \quad \text{since } \vec{r}\times\hat{r} = \emptyset. \ (\vec{r} \ \& \ \hat{r} \text{ are parallel}).$$

Then we have:

$$[H, \vec{L}] = [\frac{p^2}{2m}, \vec{L}] + [V(r), \vec{L}]$$

$$= \emptyset + \emptyset$$

$$= \emptyset$$

Problem 26.2

Note that $\tan(\tan^{-1}(x)) = x$, then taking the x-derivative of both sides gives:

$\frac{d}{dx}\tan(\tan^{-1}(x)) = 1$ (*) let $y \equiv \tan^{-1}(x)$, then we can use the chain rule:

$$\frac{d}{dx}\tan(\tan^{-1}(x)) = \frac{d}{dy}\tan(y)\cdot\frac{dy}{dx} \quad \& \quad \frac{d\tan y}{dy} = 1 + \frac{\sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

So from (*), we have:

$$1 = \frac{d}{dx}\tan(\tan^{-1}(x)) = \frac{1}{\cos^2 y}\cdot\frac{d}{dx}\tan^{-1}(x)$$

or $\frac{d}{dx}\tan^{-1}(x) = \cos^2(y)$. & $\cos^2 y + \sin^2 y = 1 \Rightarrow \tan^2 y = \frac{1}{\cos^2 y} - 1$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+\tan^2 y} = \boxed{\frac{1}{1+x^2}}$$

TOPS 35502