Problem 24.1 (Griffiths 4.13 parts a & b.)

a. In the ground state, we have:

$$\psi_{100}(r,\theta,\phi) = \sqrt{\left(\frac{2}{a}\right)^3 \frac{0!}{2\cdot(1!)^3}}\, e^{-r/a} \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

Then the expectation value of r is:

$$\langle r \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{r}{\pi a^3} e^{-2r/a} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

(integration by parts)

$$= \frac{4}{a^3}\int_0^\infty e^{-2r/a} r^3 dr = \frac{4}{a^3}\left[\left(\frac{-a}{2}\right)e^{-2r/a} r^3 \Big|_0^\infty - \int_0^\infty \left(-\frac{a}{2}\right) e^{-2r/a} 3r^2 dr\right]$$

$$= \frac{4}{a^3}\cdot\frac{3a}{2}\int_0^\infty r^2 e^{-2r/a} dr$$

$$= \frac{4}{a^3}\cdot\frac{3a}{2}\cdot\frac{2a}{2}\int_0^\infty r e^{-2r/a} dr$$

$$= \frac{6}{a}\cdot\frac{a}{2}\cdot\int_0^\infty e^{-2r/a} dr$$

$$= \boxed{\frac{3a}{2}}$$

(integration by parts - boundary terms vanish)

b. An extra factor of r will give an extra integration: $\left(\sim \frac{4r^3}{2}a\right)$

$$\langle r^2 \rangle = \boxed{3a^2}$$

c. $x = r\sin\theta\cos\phi$, & $\int_0^{2\pi}\cos\phi\, d\phi = 0$, so $\langle x \rangle = 0$

$x^2+y^2+z^2 = r^2$, & the wave function is spherically symmetric, so we expect all directions to be equivalent. Then:

$$\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle = \boxed{a^2}$$

Problem 24.2 (Griffiths 4.14)

In the ground state, $\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$

the probability of finding the particle between r_1 & r_2 is:

$$P(r_1, r_2) = \int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} \frac{1}{\pi a^3} e^{-2r/a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4}{a^3}\int_{r_1}^{r_2} e^{-2r/a} r^2 dr$$

so the probability density associated w/ the radial location is:

$$\boxed{\rho(r) = \frac{4}{a^3} e^{-2r/a} r^2}$$

Now $\frac{d\rho}{dr} = \frac{4}{a^3} e^{-2r/a}\left[2r - r^2 2/a\right] = 0 \Rightarrow \boxed{r = a}$

Problem 24.3 (Griffiths 4.15)

$$\psi_{21\pm1}(r,\theta,\phi) = \sqrt{\left(\frac{2}{2a}\right)^3 \frac{0!}{4[3!]^3}}\; e^{-r/2a}\left(\frac{2r}{2a}\right)\left[L_0^3\left(\frac{2r}{2a}\right)\right] Y_1^{\pm1}(\theta,\phi)$$

$$= \mp\sqrt{\frac{1}{a^3}\frac{1}{(6)^3}}\;\frac{e^{-r/2a}}{2}\frac{r}{a}\cdot 6\cdot\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

So

$$\psi_{211} + \psi_{21-1} = 3\left(\frac{3}{8\pi}\right)^{1/2}\frac{1}{(6a)^{3/2}}\frac{r}{a}e^{-r/2a}\sin\theta\left[e^{-i\phi} - e^{i\phi}\right]$$

$$= -i\left(\frac{3}{8\pi}\right)^{1/2}\frac{1}{\sqrt{6}\,a^{5/2}}\, r e^{-r/2a}\sin\theta\sin\phi$$

$$= -i\left(\frac{1}{8\cdot 2\pi}\right)^{1/2}\frac{1}{a^{5/2}}\, r e^{-r/2a}\sin\theta\sin\phi$$

& $\psi(\vec{r},0) = \frac{1}{\sqrt{2}}\left(\psi_{211} + \psi_{21-1}\right) = \frac{-i}{4}\frac{1}{\sqrt{2\pi}}\frac{1}{a^{5/2}}\, r e^{-r/2a}\sin\theta\sin\phi$

Both ψ_{211} & ψ_{21-1} have the same energy: $E_2 = \frac{-m}{2\hbar^2}\left[\left(\frac{q^2}{4\pi\epsilon_0}\right)^2\cdot\frac{1}{4}\right] = -\frac{\hbar^2}{2ma^2}\cdot\frac{1}{4}$.

So

$$\boxed{\psi(\vec{r},t) = \frac{-i}{4}\frac{1}{\sqrt{2\pi}}\frac{1}{a^{5/2}}\, r e^{-r/2a}\sin\theta\sin\phi\, e^{-iE_2 t/\hbar}}$$

b. $\psi^*\psi$ is time-independent, so we expect the expectation value of V to be constant:

$$\langle V\rangle = \int_0^\infty\int_0^{2\pi}\int_0^\pi \psi^*(\vec{r},0)\cdot\left(-\frac{\beta}{r}\right)\psi^*(\vec{r},0)\, r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_0^\infty\int_0^{2\pi}\int_0^\pi \frac{1}{16}\frac{1}{2\pi}\frac{1}{a^5}\, r^4 e^{-r/a}\sin^3\theta\sin^2\phi\cdot\left(-\beta/r\right) d\theta\, d\phi\, dr$$

w/ $\int_0^{2\pi}\sin^2\phi\, d\phi = \pi$ & $\int_0^\pi \sin^3\theta\, d\theta = 4/3$, so

$$= \frac{-\beta}{24a^5}\int_0^\infty r^3 e^{-r/a}\, dr \qquad \frac{1}{a^5}\int_0^\infty r^3 e^{-r/a}\, dr = \frac{+3\cdot 2}{a}\int_0^\infty e^{-r/a}\, dr = 6/a$$

(integration by parts as in first problem)

$$= \frac{-\beta}{4a}$$

w/ $\beta = \frac{q^2}{4\pi\epsilon_0}$ & $a = \frac{4\pi\epsilon_0\hbar^2}{mq^2}$ $\Rightarrow$ $\beta/a = \frac{mq^4}{(4\pi\epsilon_0)^2\hbar^2} = \frac{m}{\hbar^2}\left(\frac{q^2}{4\pi\epsilon_0}\right)^2$

then

$$\langle V\rangle = -\frac{1}{4}\frac{m}{\hbar^2}\left(\frac{q^2}{4\pi\epsilon_0}\right)^2 = \frac{1}{2}E_1 = \frac{-13.6\,\text{eV}}{2} = \boxed{-6.8\,\text{eV}}$$

TOPS.