Problem 23.1

a. $-\frac{\hbar^2}{2m}u'' + \left[\frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$ can be written as:

$$u'' - \left[\frac{m^2\omega^2}{\hbar^2}r^2 + \frac{\ell(\ell+1)}{r^2}\right]u = -\frac{2m}{\hbar^2}Eu \qquad (*)$$

& each term has units of $\frac{|u|}{|L|^2}$, so $\frac{|m||\omega|}{|\hbar|}|L| = \frac{1}{|L|} \Rightarrow \sqrt{\frac{m\omega}{\hbar}}$ has units of $\frac{1}{|L|}$

Let $z = \sqrt{\frac{m\omega}{\hbar}}\,r$, then z is dimensionless, & $\frac{d^2}{dr^2} = \frac{d^2}{\frac{\hbar}{m\omega}dz^2}$, so $(*)$ becomes:

$$\frac{m\omega}{\hbar}\frac{d^2u}{dz^2} - \left[\frac{m^2\omega^2}{\hbar^2}\cdot\frac{\hbar}{m\omega}z^2 + \frac{\ell(\ell+1)}{\frac{\hbar}{m\omega}z^2}\right]u = -\frac{2m}{\hbar^2}Eu$$

or

$$\frac{d^2u}{dz^2} - \left[z^2 + \frac{\ell(\ell+1)}{z^2}\right]u = -\frac{2}{\hbar\omega}Eu$$

let $\alpha \equiv \frac{2E}{\hbar\omega}$, then $\boxed{\frac{d^2u}{dz^2} - \left[z^2 + \frac{\ell(\ell+1)}{z^2} - \alpha\right]u = \emptyset}$ $(+)$

b. For $u = e^{-\frac{1}{2}z^2}\bar{u}$, we have $u' = -ze^{-\frac{1}{2}z^2}\bar{u} + e^{-\frac{1}{2}z^2}\bar{u}' = e^{-\frac{1}{2}z^2}(\bar{u}' - z\bar{u})$

&

$$u'' = -ze^{-\frac{1}{2}z^2}(\bar{u}' - z\bar{u}) + e^{-\frac{1}{2}z^2}(\bar{u}'' - \bar{u} - z\bar{u}')$$
$$= e^{-\frac{1}{2}z^2}(\bar{u}'' - 2z\bar{u}' + z^2\bar{u} - \bar{u})$$

& inserting this in $(+)$:

$$\bar{u}'' - 2z\bar{u}' + \cancel{z^2\bar{u}} - \bar{u} - \cancel{z^2\bar{u}} - \frac{\ell(\ell+1)}{z^2}\bar{u} + \alpha\bar{u} = \emptyset$$

gives: $\boxed{\bar{u}'' - 2z\bar{u}' + (\alpha - 1)\bar{u} - \frac{\ell(\ell+1)}{z^2}\bar{u} = \emptyset}$

c. $\bar{u} = z^p\sum_{j=0}^{\infty} c_j z^j$ gives: $\bar{u}' = z^p\sum_{j=0}^{\infty} c_j(j+p)z^{j-1}$ & $\bar{u}'' = z^p\sum_{j=0}^{\infty} c_j(j+p)(j+p-1)z^{j-2}$

putting these into the ODE for $\bar{u}$:

$$z^p\left\{\sum_{j=0}^{\infty} c_j(j+p)(j+p-1)z^{j-2} - 2\sum_{j=0}^{\infty} c_j(j+p)z^j + \sum_{j=0}^{\infty} c_j(\alpha-1)z^j - \sum_{j=0}^{\infty} c_j\,\ell(\ell+1)z^{j-2}\right\} = \emptyset$$

or:

$$\sum_{j=0}^{\infty} c_j\left[(j+p)(j+p-1) - \ell(\ell+1)\right]z^{j-2} + \sum_{j=0}^{\infty} c_j\left[(\alpha-1) - 2(j+p)\right]z^j = \emptyset$$

let $k = j-2$

$$\sum_{k=-2}^{\infty} c_{k+2}\left[(k+2+p)(k+1+p) - \ell(\ell+1)\right]z^k + \sum_{j=0}^{\infty} c_j\left[(\alpha-1) - 2(j+p)\right]z^j = \emptyset$$

and finally, we have:

$$c_0\left[p(p-1) - \ell(\ell+1)\right]z^{-2} + c_1\left[p(p+1) - \ell(\ell+1)\right]z^{-1} +$$
$$+ \sum_{j=0}^{\infty}\left\{c_j\left[(\alpha-1) - 2(j+p)\right] + c_{j+2}\left[(j+2+p)(j+1+p) - \ell(\ell+1)\right]\right\}z^j = \emptyset$$

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Problem 23.1 (continued)

From the first two terms, we get: $\rho = -\ell, \ell+1$ (from z^{-2} term) and $\rho = \ell, -\ell-1$. Take $\rho = \ell+1$, then the recursion relation is:

$$C_{j+2} = -\frac{(\alpha-1)+2(j+\ell+1)}{(j+\ell+2)(j+\ell+3)-\ell(\ell+1)}\, C_j$$

& if we insist that the series truncate, then:

$$\alpha - 1 - 2(j+\ell+1) = \varnothing \Rightarrow \alpha = 3+2(j+\ell)$$

& $\alpha \equiv \frac{2E}{\hbar\omega} = 3 + 2(j+\ell) \Rightarrow E = \left[\frac{3}{2} + (j+\ell)\right]\hbar\omega$

we can let $n \equiv j+\ell$, then $\boxed{E_n = \left(\frac{3}{2}+n\right)\hbar\omega}$

Problem 23.2

Starting from: $-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + \frac{1}{2}m\omega^2(x^2+y^2+z^2)\psi(x,y,z)$

separation of variables will give: $\psi(x,y,z) = X(x)Y(y)Z(z)$

$$-\frac{\hbar^2}{2m}\frac{X''}{X} - \frac{\hbar^2}{2m}\frac{Y''}{Y} - \frac{\hbar^2}{2m}\frac{Z''}{Z} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2 z^2 = E$$

or:

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega^2 x^2\right]}_{=E_x} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{Y''}{Y} + \frac{1}{2}m\omega^2 y^2\right]}_{=E_y} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{Z''}{Z} + \frac{1}{2}m\omega^2 z^2\right]}_{=E_z} = E$$

we have 3 copies of the one dimensional harmonic oscillator, & we know that:

$$E_x = \left(\tfrac{1}{2}+n_x\right)\hbar\omega,\quad E_y = \left(\tfrac{1}{2}+n_y\right)\hbar\omega,\quad E_z = \left(\tfrac{1}{2}+n_z\right)\hbar\omega$$

for integers $n_x, n_y, \& \ n_z$, so

$$E = E_x + E_y + E_z = \boxed{\left(\tfrac{3}{2} + (n_x+n_y+n_z)\right)\hbar\omega}$$

& let $n \equiv n_x + n_y + n_z$ ($n = \varnothing \rightarrow \infty \in \mathbb{Z}$), then:

$\boxed{E_n = \left(\frac{3}{2}+n\right)\hbar\omega}$ as above