Problem 22.1

a. For $\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{2mr^2}{\hbar^2}(V(r)-E) = l(l+1)$, take $u = rf(r)$,

then: $\frac{df}{dr} = \frac{d}{dr}\left(\frac{u}{r}\right) = -\frac{1}{r^2}u + u'/r$, so our radial eqn. becomes:

$$\frac{r}{u}\frac{d}{dr}(-u+u'r) - \frac{2mr^2}{\hbar^2}(V-E) = l(l+1)$$

$$\frac{r}{u}\left[-\cancel{u'}+\cancel{u'}+u''r\right] - \frac{2mr^2}{\hbar^2}(V-E) = l(l+1)$$

or:

$$u'' - \left[\frac{2m}{\hbar^2}(V-E) + \frac{l(l+1)}{r^2}\right]u = \emptyset$$

as desired.

b. For $V(r) = -\beta/r$ w/ $|\beta| = ML^3T^{-2}$

$$\underbrace{u''}_{u/L^2} - \left[\underbrace{-\frac{2m}{\hbar^2}\frac{\beta}{r}}_{=1/L^2} - \underbrace{\frac{2mE}{\hbar^2}}_{1/L^2} + \underbrace{\frac{l(l+1)}{r^2}}_{1/L^2}\right]u = \emptyset. \quad (*)$$

to get $\frac{1}{L^2} = \left|\frac{m}{\hbar^2}\frac{\beta}{r}\right|$ we have: $\left|\frac{m\beta}{\hbar^2}\right| \sim \frac{1}{L} \Rightarrow \left|\frac{\hbar^2}{m\beta}\right| \sim L$

Take $z = \frac{m\beta}{\hbar^2}\cdot r$, then z is unitless. & $dz = \frac{m\beta}{\hbar^2}dr \Rightarrow \frac{d}{dr} = \frac{m\beta}{\hbar^2}\frac{d}{dz}$, then $(*)$ is:

$$\left(\frac{m\beta}{\hbar^2}\right)^2\frac{d^2u}{dz^2} - \left[-\frac{2m^2\beta^2}{\hbar^4}\frac{1}{z} - \frac{2mE}{\hbar^2} + \frac{l(l+1)}{z^2}\cdot\frac{m^2\beta^2}{\hbar^4}\right]u = \emptyset$$

so

$$\frac{d^2u}{dz^2} - \left[-\frac{2}{z} - \frac{2\hbar^2}{m\beta^2}E + \frac{l(l+1)}{z^2}\right]u = \emptyset. \Rightarrow \boxed{\frac{d^2u}{dz^2} - \left[-\frac{2}{z} + \alpha^2 + \frac{l(l+1)}{z^2}\right]u = \emptyset}$$

& let $\alpha^2 = -\frac{2\hbar^2}{\beta^2}E$

c. For z large, we have: $\frac{d^2u}{dz^2} - \alpha^2 u = \emptyset \Rightarrow u(z) = Ae^{\alpha z} + Be^{-\alpha z}$

& let $G(z) = e^{-\alpha z}$

d. $u(z) = \bar{u}\cdot e^{-\alpha z}$, then $u' = \bar{u}'e^{-\alpha z} - \alpha\bar{u}e^{-\alpha z} = e^{-\alpha z}(\bar{u}' - \alpha\bar{u})$

$u'' = -\alpha e^{-\alpha z}(\bar{u}' - \alpha\bar{u}) + e^{-\alpha z}(\bar{u}'' - \alpha\bar{u}') = e^{-\alpha z}(\bar{u}'' - 2\alpha\bar{u}' + \alpha^2\bar{u})$

and inserting this in the result from part b.:

$$\bar{u}'' - 2\alpha\bar{u}' + \alpha^2\bar{u} + \left(2/z - \alpha^2 - l(l+1)/z^2\right)\bar{u} = \emptyset.$$

or

$$\boxed{\bar{u}'' - 2\alpha\bar{u}' + \left(2/z - l(l+1)/z^2\right)\bar{u} = \emptyset}$$

e. Let $\bar{u}(z) = z^p\sum_{j=0}^{\infty} a_j z^j$, then: $\bar{u}' = z^p\sum_{j=0}^{\infty} a_j(j+p)z^{j-1}$ & $\bar{u}'' = z^p\sum_{j=0}^{\infty} a_j(j+p)(j+p-1)z^{j-2}$

The ODE from part d. reads:

$$z^p\left\{\sum_{j=0}^{\infty} a_j(j+p)(j+p-1)z^{j-2} - 2\alpha\sum_{j=0}^{\infty} a_j(j+p)z^{j-1} + 2\sum_{j=0}^{\infty} a_j z^{j-1} - l(l+1)\sum_{j=0}^{\infty} a_j z^{j-2}\right\} = \emptyset.$$

or:

$$\sum_{j=0}^{\infty}\left[(j+p)(j+p-1) - l(l+1)\right]a_j z^{j-2} + 2\sum_{j=0}^{\infty}\left(1-\alpha(j+p)\right)a_j z^{j-1} = \emptyset$$

re-labelling the sums, we have: (set $j-2=k-1$ in the first term, and $j\to k$ in the second)

$$\sum_{k=0}^{\infty}\left[\left[(k+p+1)(k+p) - l(l+1)\right]a_{k+1} + 2(1-\alpha(k+p))a_k\right]z^k + \underbrace{\left[p(p-1) - l(l+1)\right]a_0 z^{-2}}_{\text{the } k=-1 \text{ term from the first sum.}} = \emptyset$$

We have, killing all orders in z:

$$\rho(\rho-1) - l(l+1) = \varnothing \quad \& \quad a_{k+1} = \frac{-2(1-\alpha(k+\rho))}{(k+\rho+1)(k+\rho) - l(l+1)}\, a_k$$

the first equation has $\rho = l+1$ or $\rho = -l$.
But $\rho = -l$ gives a leading order term: $\bar{u}(z) = z^{-l}$, & this blows up at $z = \varnothing$, so

$$\boxed{\rho = l+1}$$

then our recursion reads

f.

$$a_{k+1} = \frac{-2(1-\alpha(k+l+1))}{(k+l+2)(k+l+1) - l(l+1)}\, a_k = \frac{2\alpha(k+l+1)-2}{(k+1)(k+2(l+1))}\, a_k.$$

Assuming the series truncates, there must be an a_J such that $a_{J+1} = \varnothing$ so

$$2\alpha(J+l+1) - 2 = \varnothing \implies \alpha = \frac{1}{J+l+1}$$

& $\quad \alpha^2 = -\frac{2\hbar^2}{m\beta^2}\cdot E = \frac{1}{(J+l+1)^2} \implies E = -\frac{m\beta^2}{2\hbar^2 (J+l+1)^2}$

& let $n \equiv J+l+1$, then $\boxed{E_n = \frac{-m\beta^2}{2\hbar^2 n^2}}$