Problem 2.1

Given $\sin\alpha = \frac{1}{2i}\left[e^{i\alpha} - e^{-i\alpha}\right]$

a. $\int_0^d \sin\left(\frac{j\pi x}{d}\right)\sin\left(\frac{k\pi x}{d}\right)dx = -\frac{1}{4}\int_0^d \left[e^{ij\pi x/d} - e^{-ij\pi x/d}\right]\left[e^{ik\pi x/d} - e^{-ik\pi x/d}\right]dx$

$$= -\frac{1}{4}\int_0^d \left[e^{i(j+k)\pi x/d} - e^{i(j-k)\pi x/d} - e^{i(k-j)\pi x/d} + e^{-i(j+k)\pi x/d}\right]dx$$

so we have four integrals to evaluate.

$$\int_0^d e^{i(j+k)\pi x/d}dx = \frac{d}{i(j+k)\pi}e^{i(j+k)\pi x/d}\Big|_{x=0}^{d} = \frac{d}{i(j+k)\pi}\left(e^{i(j+k)\pi} - 1\right)$$

$$\int_0^d e^{i(j-k)\pi x/d}dx = \frac{d}{i(j-k)\pi}e^{i(j-k)\pi x/d}\Big|_{x=0}^{d} = \frac{d}{i(j-k)\pi}\left(e^{i(j-k)\pi} - 1\right)$$

$$\int_0^d e^{i(k-j)\pi x/d}dx = \frac{d}{i(k-j)\pi}e^{i(k-j)\pi x/d}\Big|_{x=0}^{d} = \frac{d}{i(k-j)\pi}\left(e^{i(k-j)\pi} - 1\right)$$

$$\int_0^d e^{-i(j+k)\pi x/d}dx = \frac{-d}{i(j+k)\pi}e^{-i(j+k)\pi x/d}\Big|_{x=0}^{d} = -\frac{d}{i(j+k)\pi}\left(e^{-i(j+k)\pi} - 1\right)$$

putting these together, we have:

$$\int_0^d \sin\left(\frac{j\pi x}{d}\right)\sin\left(\frac{k\pi x}{d}\right)dx = -\frac{1}{4}\frac{d}{i\pi}\left\{\frac{1}{j+k}\left(e^{i(j+k)\pi} - e^{-i(j+k)\pi}\right) - \frac{1}{j-k}\left(e^{i(j-k)\pi} - e^{-i(j-k)\pi}\right)\right\}$$

$$= -\frac{1}{4}\frac{d}{i\pi}\left\{\frac{1}{j+k}\cdot 2i\sin((j+k)\pi) - \frac{1}{j-k}2i\sin((j-k)\pi)\right\}$$

and note that $\sin((j+k)\pi) = \varnothing$ since $j+k$ is an integer, & the same is true for $\sin((j-k)\pi) = \varnothing$, so

$$= \varnothing.$$

b. $\int_0^d \sin^2\left(\frac{j\pi x}{d}\right)dx = -\frac{1}{4}\int_0^d \left[e^{ij\pi x/d} - e^{-ij\pi x/d}\right]\left[e^{ij\pi x/d} - e^{-ij\pi x/d}\right]dx$

$$= -\frac{1}{4}\int_0^d \left[e^{2ij\pi x/d} - 2 + e^{-2ij\pi x/d}\right]dx$$

& $\int_0^d e^{2ij\pi x/d}dx = \frac{d}{2\pi ij}e^{2ij\pi x/d}\Big|_{x=0}^{d} = \frac{d}{2\pi ij}\left(e^{2\pi ij} - 1\right)$

$\int_0^d e^{-2ij\pi x/d}dx = \frac{-d}{2\pi ij}e^{-2ij\pi x/d}\Big|_{x=0}^{d} = -\frac{d}{2\pi ij}\left(e^{-2\pi ij} - 1\right)$

so $\int_0^d \sin^2\left(\frac{j\pi x}{d}\right)dx = -\frac{1}{4}\left[-2d + \frac{d}{2\pi ij}\left(e^{2\pi ij} - e^{-2\pi ij}\right)\right]$

$$= \frac{d}{2} - \frac{d}{8\pi ij}\cdot 2i\underbrace{\sin(2\pi j)}_{=\varnothing}$$

$$= d/2$$

So that $2/d\int_0^d \sin^2\left(\frac{j\pi x}{d}\right)dx = 1$.

Problem 2.2

a. $\vec{v} = 3\hat{x} + \hat{y}$, $\vec{w} = -2\hat{x} + 6\hat{y}$ has: $\boxed{\vec{v}\cdot\vec{w} = -6+6 = \emptyset}$

we can normalize via: $\hat{v} = \frac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}} = \frac{3}{\sqrt{10}}\hat{x} + \frac{1}{\sqrt{10}}\hat{y}$

$$\hat{w} = \frac{\vec{w}}{\sqrt{\vec{w}\cdot\vec{w}}} = -\frac{2}{\sqrt{40}}\hat{x} + \frac{6}{\sqrt{40}}\hat{y}$$

b. $\vec{a} = 10\hat{x} - 3\hat{y}$, & we want $\vec{a} = \_\,\hat{v} + \_\_\,\hat{w}$, but we know that:

$$\vec{a} = (\vec{a}\cdot\hat{v})\hat{v} + (\vec{a}\cdot\hat{w})\hat{w}$$

so the relevent coefficients are: $\vec{a}\cdot\hat{v} = 3\sqrt{10} - 3/\sqrt{10}$

$$\vec{a}\cdot\hat{w} = \frac{-20}{\sqrt{40}} - 18/\sqrt{40} = -\sqrt{10} - 9/\sqrt{10}$$

so

$$\vec{a} = \left[3\sqrt{10} - 3/\sqrt{10}\right]\hat{v} + \left[-\sqrt{10} - 9/\sqrt{10}\right]\hat{w} = \boxed{\frac{27}{\sqrt{10}}\hat{v} - \frac{19}{\sqrt{10}}\hat{w}}$$

Problem 2.3 (Griffiths A.1)

a. $\boxed{\text{Yes}}$ vectors of the form $a = a_x\hat{i} + a_y\hat{j} + \emptyset\cdot\hat{k}$ can be added together & multiplied by numbers w/out changing the $\hat{k}$ coefficient. This is a $\boxed{\text{2-dimensional}}$ space.

b. $\boxed{\text{No}}$ Take $a = \hat{x} + \hat{y} + \hat{z}$ & $b = 2\hat{x} + 2\hat{y} + \hat{z}$, then

$a + b = 3\hat{x} + 3\hat{y} + 2\hat{z}$, & this is not in the space, since the coefficient of $\vec{z}$ is not 1.

c. Take $\vec{a} = a(\hat{x} + \hat{y} + \hat{z})$ & $\vec{b} = b(\hat{x} + \hat{y} + \hat{z})$, then the space is closed under addition, since:

$$\vec{a} + \vec{b} = (a+b)(\hat{x} + \hat{y} + \hat{z}) \quad \text{is in the space}$$

& $\alpha\vec{a} = (\alpha a)(\hat{x} + \hat{y} + \hat{z})$ is also in the space.

the space contains the zero vector, so it $\boxed{\text{is}}$ a vector space. (of dimension 1)