Problem 19.1

a. $\mathbb{A} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the vectors $\vec{v}_1 \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{v}_2 \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors:

$$\mathbb{A}\vec{v}_1 = \vec{v}_1 \quad \& \quad \mathbb{A}\vec{v}_2 = \vec{v}_2 \quad \text{w/ eigenvalue } 1.$$

Those are not eigenvectors of $\mathbb{B} \doteq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

$$\mathbb{B}\vec{v}_1 \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbb{B}\vec{v}_2 \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b. $$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \emptyset \Rightarrow \lambda = \emptyset,\ \lambda = 2$$

the eigenvectors are: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \emptyset \Rightarrow b = -a$, so $\vec{w}_1 \doteq \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

& $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a + b = 2a \Rightarrow a = b$
so $\vec{w}_2 \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

& $\vec{w}_1, \vec{w}_2$ are clearly eigenvectors of $\mathbb{A}$ as well.

Problem 19.2

We have: $$\langle Q \rangle = \int_{-\infty}^{+\infty} \Psi^*(x)\, \hat{Q}\, \Psi(x)\, dx$$
$$\overset{!}{=} \int_{-\infty}^{+\infty} (\hat{Q}^+ \Psi(x))^* \Psi(x)\, dx$$
$$\overset{!}{=} -\int_{-\infty}^{+\infty} (\hat{Q}\Psi(x))^* \Psi(x)\, dx$$
$$\overset{!}{=} -\left[\int_{-\infty}^{+\infty} \hat{Q}\Psi(x)\, \Psi(x)^*\right]^*$$
$$\overset{!}{=} -\langle Q \rangle^*$$

and a number whose complex conjugate is equal to its negative is purely imaginary.

Problem 19.3

a. $|\Psi(t)\rangle = \int_{-\infty}^{+\infty} \Phi(k,t)\,|\Psi_k\rangle\,dk$

If we insert this solution in $\hat{H}|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle$, we get:

$$\hat{H}|\Psi(t)\rangle = \int_{-\infty}^{+\infty}\Phi(k,t)\,E(k)\,|\Psi_k\rangle\,dk = i\hbar\int_{-\infty}^{+\infty}\frac{\partial\Phi(k,t)}{\partial t}\,|\Psi_k\rangle\,dk$$

Hitting both sides w/ $\langle\Psi_{k'}|$:

$$\langle\Psi_{k'}|\hat{H}|\Psi(t)\rangle = \int_{-\infty}^{+\infty}\Phi(k,t)\,E(k)\underbrace{\langle\Psi_{k'}|\Psi_k\rangle}_{\delta(k-k')}dk = i\hbar\int_{-\infty}^{+\infty}\frac{\partial\Phi(k,t)}{\partial t}\underbrace{\langle\Psi_{k'}|\Psi_k\rangle}_{\delta(k-k')}dk$$

& so, using the delta functions to eliminate the integrals:

$$\Phi(k',t)\,E(k') = i\hbar\frac{\partial\Phi(k',t)}{\partial t}.$$

This has solution:

$$\Phi(k',t) = f(k')\,e^{-iE(k')t/\hbar}$$

where $f(k')$ is an arbitrary function of k'.