Problem 18.1

a. $H\Psi(p,t) = i\hbar \frac{\partial \Psi(p,t)}{\partial t}$ w/ $H(-\frac{\hbar}{i}\frac{d}{dx}, p) = \frac{p^2}{2m} + F\frac{\hbar}{i}\frac{\partial}{\partial p}$

gives:

$$\boxed{\frac{p^2}{2m}\Psi(p,t) + F\frac{\hbar}{i}\frac{\partial \Psi}{\partial p} = i\hbar\frac{\partial \Psi}{\partial t}} \qquad (*)$$

b. Let $\Psi(p,t) = P(p-Ft)\,Q(p)$, then

$$\frac{\partial \Psi}{\partial p} = P'(p-Ft)Q(p) + P(p-Ft)Q'(p)$$

& $$\frac{\partial \Psi}{\partial t} = -FP'(p-Ft)Q(p)$$

so that $(*)$ reads: $\frac{p^2}{2m}P(p-Ft)Q - i\hbar F[\cancel{P'Q} + PQ'] = -i\hbar F \cancel{P'Q}$

& then: $\frac{p^2}{2m}PQ - i\hbar F P Q' = \emptyset$

$$\Downarrow$$

$$\frac{dQ}{dp} = \frac{p^2}{2mi\hbar F}Q = -\frac{ip^2}{2m\hbar F}Q.$$

we can solve this equation: $\frac{1}{Q}dQ = \frac{-ip^2}{2m\hbar F}dp$

$$\log(Q) = -\frac{ip^3}{6m\hbar F} + C$$

so that $\boxed{Q(p) = Q_0 e^{-\frac{ip^3}{6m\hbar F}}}$.

c. $\Psi(p,t) = P(p-Ft)e^{\frac{-ip^3}{6m\hbar F}}$ (letting P soak up the constant Q_0).

& $\Psi^*(p,t)\Psi(p,t) = P(p-Ft)^* P(p-Ft)$

so we must have: $\boxed{\int_{-\infty}^{+\infty} |P(p-Ft)|^2 dp = 1.}$

d. $\langle p\rangle = \int_{-\infty}^{+\infty} \Psi^*(p,t)\, p\, \Psi(p,t)\,dp = \int_{-\infty}^{+\infty} P^*(p-Ft)\, p\, P(p-Ft)\,dp$

let $z \equiv p - Ft$, $dz = dp$, then

$$= \underbrace{\int_{-\infty}^{+\infty} P^*(z)\, z\, P(z)\,dz}_{\emptyset \text{ since at } t=\emptyset,\ \langle p\rangle = \emptyset} + Ft\int_{-\infty}^{+\infty} P^*(z)P(z)\,dz$$

$$= Ft$$

This is also clear from Ehrenfest's theorem: $\frac{d\langle p\rangle}{dt} = \langle -\frac{dV}{dx}\rangle = \langle -(-F)\rangle = F$

so $\langle p\rangle = Ft$ w/ $\langle p\rangle|_{t=0} = \emptyset$.

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Problem 18.2

a. We want to find $f(x)$ such that

$$Pf(x) = \alpha f(x) \qquad w/ \ Pf(x) = f(-x)$$

so that $f(-x) = \alpha f(x)$ – if we hit both sides of the equation w/ P:

$$\underbrace{Pf(-x)}_{f(x)} = \alpha \underbrace{Pf(x)}_{\parallel}$$

$$f(x) = \alpha^2 f(x)$$

Then we learn that $\alpha^2 = 1$, so that $\alpha = \pm 1$. Take $\alpha = 1$, then

$$Pf(x) = 1 \cdot f(x) \Rightarrow f(-x) = f(x)$$ & we see that even functions are eigenfunctions

For $\alpha = -1$, $$Pf(x) = -f(x) \Rightarrow f(-x) = -f(x)$$ so odd functions are also eigenfunctions.

b. $$\int_{-\infty}^{+\infty} f(x)^* P g(x) dx = \int_{-\infty}^{+\infty} f(x)^* g(-x) dx$$ let $y = -x$, then

$$= -\int_{\infty}^{-\infty} f(-y)^* g(y) dy$$

$$= \int_{-\infty}^{+\infty} [Pf(y)]^* g(y) dy$$

& so P is Hermitian.

c. If $f(x)$ is an eigenfunction of P w/ e-val $+1$, so that

$$Pf(x) = f(x) \Rightarrow f(-x) = f(x), \text{ an even function}$$

Take $g(x)$ to be an eigenfunction of P w/ e-val -1,

$$Pg(x) = -g(x) \Rightarrow g(-x) = -g(x), \text{ an odd function}$$

Then $\int_{-\infty}^{+\infty} f(x)^* g(x) dx = \emptyset$ since the product $f(x)^* g(x)$ is odd, & the limits of integration allow x to run over all negative & positive values.

Problem 18.3

$$\int_{-\infty}^{+\infty} f(x)^* a_+ g(x) dx = \frac{1}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{+\infty} f(x)^* \left[-\hbar \frac{dg}{dx} + m\omega x g\right] = \frac{1}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{+\infty} \left[\hbar \frac{df^*}{dx} + f^* m\omega x\right] g dx$$

$$= \frac{1}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{+\infty} \left[\hbar \frac{df}{dx} + f m\omega x\right]^* g dx$$

$$= \int_{-\infty}^{+\infty} [a_- f]^* g dx$$

w/ $a_- = \frac{1}{\sqrt{2m\hbar\omega}}\left[\hbar \frac{d}{dx} + m\omega x\right]$, so $\boxed{a_+^\dagger = a_-}$