Problem 16.2

a. For $f_n(x) = \frac{1}{2\pi}\int_{-n}^{+n} e^{ik(x-a)}\,dk$, we can perform the integration:

$$= \frac{1}{2\pi}\frac{1}{i(x-a)}\left(e^{in(x-a)} - e^{-in(x-a)}\right)$$

$$= \frac{1}{\pi}\frac{\sin(n(x-a))}{(x-a)}$$

Then $\int_{-\infty}^{+\infty} f_n(x)dx = \int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{\sin(n(x-a))}{(x-a)}dx$ let $y = n(x-a)$, then $dy = ndx$, &

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(y)}{y/n}\cdot\frac{1}{n}dy = \frac{1}{\pi}\cdot 2\int_0^{\infty}\frac{\sin(y)}{y}dy = 1 \checkmark$$

b. For $x-a \equiv \epsilon$, $f_n(x) = \frac{1}{\pi}\frac{\sin(n\epsilon)}{\epsilon}$ & as we send $\epsilon \to \emptyset$, we can send $n \to \infty$ so that $n\cdot\epsilon$ is fixed, then

$$\frac{1}{\pi}\frac{\sin(n\epsilon)}{\epsilon} \xrightarrow[\epsilon\downarrow\emptyset]{n\uparrow\infty} \frac{1}{\pi}\frac{\text{const.}}{\emptyset} = \infty$$

Problem 16.3

We want $f_\alpha(x)$ w/ $a_- f_\alpha(x) = \alpha f_\alpha(x)$. & $a_- = \frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)$

so:

$$\sqrt{\frac{\hbar}{2m\omega}}\, f'_\alpha(x) + \sqrt{\frac{m\omega}{2\hbar}}\, x f_\alpha(x) = \alpha f_\alpha(x)$$

& we can write this equation as:

$$\frac{f'_\alpha(x)}{f_\alpha(x)} = \left[\sqrt{\frac{2m\omega}{\hbar}}\,\alpha - \frac{m\omega}{\hbar}x\right]$$

Integrating both sides, we get

$$\log(f_\alpha(x)) = \sqrt{\frac{2m\omega}{\hbar}}\,\alpha x - \frac{m\omega}{2\hbar}x^2$$

so

$$\boxed{f_\alpha(x) = \underbrace{\ }_{\text{can put an arbitrary constant out front}} e^{\sqrt{\frac{2m\omega}{\hbar}}\alpha x - \frac{m\omega}{2\hbar}x^2}}$$

Problem 16.4 (Griffiths 3.2)

a. We require that $\int_0^1 f(x)^* f(x)\,dx$ exist, for $f(x) = x^\nu$, we have:

$$\int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1} x^{2\nu+1}\Big|_{x=0}^{1} = \frac{1}{2\nu+1}$$

& we want $2\nu+1 > \emptyset \Rightarrow \boxed{\nu > -\tfrac{1}{2}}$

b. If $\nu = \tfrac{1}{2}$, $\int_0^1 f(x)^* f(x)\,dx = \int_0^1 x\,dx = \tfrac{1}{2}$, (& $\tfrac{1}{2} > -\tfrac{1}{2}$) so $f(x) = \sqrt{x}$ is in the Hilbert space.

$\int_0^1 (x f(x))^* (x f(x))\,dx = \int_0^1 x^3\,dx = \tfrac{1}{4}$ $(<\infty)$, & $x f(x)$ is also in the Hilbert space

$\frac{d}{dx}\sqrt{x} = \tfrac{1}{2} x^{-1/2}$ & here, $\nu = -\tfrac{1}{2} \not> -\tfrac{1}{2}$, so $\frac{d}{dx} f(x)$ is not in the Hilbert space.

Problem 16.5 (Griffiths 3.7)

a. We have: $\hat{Q} f(x) = q f(x)$, $\hat{Q} g(x) = q g(x)$. Take a generic linear combination:

$$\hat{Q}[A f(x) + B g(x)] = A \underbrace{\hat{Q} f(x)}_{= q f(x)} + B \underbrace{\hat{Q} g(x)}_{= q g(x)} = q[A f(x) + B g(x)]$$

so $A f(x) + B g(x)$ is also an eigenfunction w/ eigenvalue q.

b. For $f(x) = e^x$, $\frac{d^2}{dx^2} f(x) = e^x = f(x)$, so $f(x)$ is an eigenfunction of $\frac{d^2}{dx^2}$ w/ eigenvalue 1, & similarly:

$$\frac{d^2}{dx^2} g(x) = -\frac{d}{dx} e^{-x} = e^{-x} = g(x)$$

, so $g(x)$ is also an eigenfunction w/ eigenvalue 1.

let $h(x) = A f(x) + B g(x)$, & $j(x) = C f(x) + D g(x)$, then

$$\int_{-1}^{1} h^*(x) j(x)\,dx = \int_{-1}^{1} (Af + Bg)(Cf + Dg)\,dx$$
$$= \int_{-1}^{1} [AC e^{2x} + AD + BC + BD e^{-2x}]\,dx$$
$$= AD + BC + AC \cdot \tfrac{1}{2}(e^2 - e^{-2}) - BD \cdot \tfrac{1}{2}(e^{-2} - e^2)$$
$$= (AD + BC) + \tfrac{1}{2}(AC + BD)(e^2 - e^{-2})$$

to get zero, we must have: $AD + BC = \emptyset$ & $AC + BD = \emptyset$

& these can be solved w/: $B = \pm A$, $D = \mp C$, so

$$\boxed{\begin{aligned} h(x) &= A(e^x + e^{-x}) = 2A\cosh(x) \\ j(x) &= C(e^x - e^{-x}) = 2C\sinh(x) \end{aligned}}$$

or vice-versa (for the other sign choice)

The remaining coefficients, A & C, can be used to set normalization.