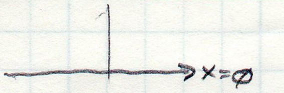
Problem 15.1

For $V(x) = \alpha x$



we will have; $\psi(x) = \begin{cases} \psi_I(x) & x \leq \emptyset \\ \psi_{II}(x) & x \geq \emptyset \end{cases}$ satisfying Schrödinger's eqn. on the left & right.

the wave function will be continuous: $\psi_I(\emptyset) = \psi_{II}(\emptyset)$ at the delta spike location, & have discontinuity in its derivative:

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=\emptyset} - \left.\frac{d\psi_I}{dx}\right|_{x=\emptyset} = \frac{2m\alpha}{\hbar^2}\psi_I(\emptyset).$$

On the left: $-\frac{\hbar^2}{2m}\psi_I'' = E\psi_I$ & $E > \emptyset$ to get bounded solutions.
let $k^2 = \frac{2mE}{\hbar^2}$, then:

$$\psi_I'' = -k^2\psi_I \Rightarrow \psi_I(x) = \underbrace{Ae^{ikx}}_{\text{right travelling}} + \underbrace{Be^{-ikx}}_{\text{left travelling}}$$

similarly, on the right we will have: $\psi_{II}(x) = Fe^{ikx}$, where we omit the left-travelling solution, specializing to the "scattering" setup.

continuity:

$$\psi_I(\emptyset) = \psi_{II}(\emptyset)$$
$$A + B = C$$

& deriv. discontinuity:

$$\psi_{II}'(\emptyset) - \psi_I'(\emptyset) = \frac{2m\alpha}{\hbar^2}\psi_I(\emptyset)$$
$$ik(\underset{A+B}{C}) - ik(A-B) = \frac{2m\alpha}{\hbar^2}(A+B)$$

so

$$2B = -i\frac{2m\alpha}{k\hbar^2}(A+B), \quad \text{let } \beta \equiv \frac{2m\alpha}{k\hbar^2},$$

$$B(1+i\beta) = -i\beta A \Rightarrow \boxed{B = \frac{-i\beta}{1+i\beta}A}$$

& $C = A + B = A - \frac{i\beta}{1+i\beta}A = \frac{1}{1+i\beta}A$

The reflection coefficient is: $\boxed{R \equiv \frac{B^*B}{A^*A} = \frac{\beta^2}{1+\beta^2}}$

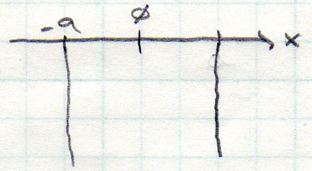
$\boxed{T = \frac{C^*C}{A^*A} = \frac{1}{1+\beta^2}}$ $\Big\rangle$ $R+T=1$

(note that R & T here are identical to the delta well).

Problem 15.2 (Griffiths 2.27)

For $V(x) = -\alpha[\delta(x+a)+\delta(x-a)]$

a. The potential looks like:



b. To find the bound states, we'll solve the time-independent Schrödinger eqn. starting w/:

$$\psi(x)=\begin{cases}\psi_I(x) & x<-a\\ \psi_{II}(x) & -a<x<a \\ \psi_{III}(x) & x>a\end{cases}$$

So we need to solve: $-\frac{\hbar^2}{2m}\psi_I'' = E\psi_I$, $-\frac{\hbar^2}{2m}\psi_{II}''=E\psi_{II}$, & $-\frac{\hbar^2}{2m}\psi_{III}''=E\psi_{III}$ w/ $E<0$.

then impose continuity: $\psi_I(-a)=\psi_{II}(-a)$, $\psi_{II}(a)=\psi_{III}(a)$

and derivative dis-continuity:

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=-a} - \left.\frac{d\psi_I}{dx}\right|_{x=-a} = -\frac{2m\alpha}{\hbar}\psi_I(-a) \;\&\; \left.\frac{d\psi_{III}}{dx}\right|_{x=a} - \left.\frac{d\psi_{II}}{dx}\right|_{x=a} = -\frac{2m\alpha}{\hbar^2}\psi_{III}(a).$$

let $k^2 = \frac{2m|E|}{\hbar^2}$ then in regions I & III, we have:

$\psi_I(x)=Ae^{kx}+Be^{-kx}$ w/ $B=0$ to avoid exp. growth in $x<0$.

$\psi_{III}(x)=Fe^{kx}+Ge^{-kx}$ w/ $F=0$ to avoid growth in $x>0$.

In region II, we retain e^{kx} & e^{-kx}, so $\psi_{II}(x)=Ce^{kx}+De^{-kx}$

continuity: $\psi_I(-a)=\psi_{II}(-a) \Rightarrow Ae^{-ka}=Ce^{-ka}+De^{ka}$

$\psi_{II}(a)=\psi_{III}(a)\Rightarrow Ge^{-ka}=Ce^{+ka}+De^{-ka}$

Derivative discontinuity: $\left.\frac{d\psi_{II}}{dx}\right|_{x=-a}-\left.\frac{d\psi_I}{dx}\right|_{x=-a}=-\frac{2m\alpha}{\hbar^2}\psi_I(-a)$

$\hookrightarrow k(Ce^{-ka}-De^{+ka})-kAe^{-ka}=-\frac{2m}{\hbar^2}\alpha Ae^{-ka}$ (*)

$\left.\frac{d\psi_{III}}{dx}\right|_{x=a}-\left.\frac{d\psi_{II}}{dx}\right|_{x=a}=-\frac{2m\alpha}{\hbar^2}\psi_{II}(a).$

$\hookrightarrow -kGe^{-ka}-k(Ce^{ka}-De^{-ka})=-\frac{2m\alpha}{\hbar^2}Ge^{-ka}$ (+)

we can rewrite (*) as: $Ce^{-ka}-De^{ka}=(1-\frac{2m\alpha}{k\hbar^2})(Ce^{-ka}+De^{ka})$ using $Ae^{-ka}=Ce^{-ka}+De^{k}$

& (+) as $-(Ce^{ka}-De^{-ka})=(1-\frac{2m\alpha}{k\hbar^2})(Ce^{ka}+De^{-ka})$

let $\gamma=1-\frac{2m\alpha}{k\hbar^2}$, then we have $Ce^{-ka}(1-\gamma)=De^{ka}(1+\gamma)\Rightarrow D=Ce^{-2ka}\frac{(1-\gamma)}{1+\gamma}$

& $Ce^{ka}(-1-\gamma)=De^{-ka}(-1+\gamma)=-Ce^{-3ka}\frac{(1-\gamma)^2}{(1+\gamma)}$

or $e^{4ka}=\frac{(1-\gamma)^2}{(1+\gamma)^2}$

Problem 15.2 (continued)

We have $1-\gamma = 1-\left(1-\frac{2m\alpha}{k\hbar^2}\right) = \frac{2m\alpha}{k\hbar^2}$

$$1+\gamma = 1+\left(1-\frac{2m\alpha}{k\hbar^2}\right) = 2\left(1-\frac{m\alpha}{k\hbar^2}\right)$$

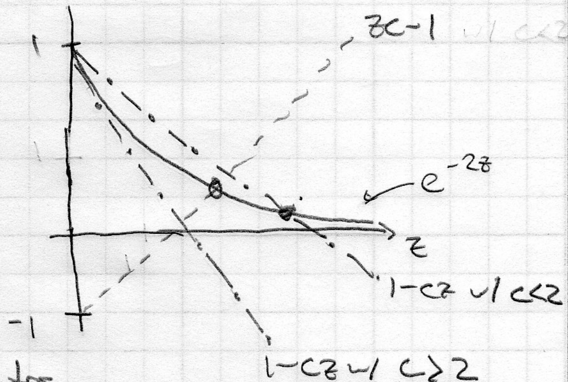
so

$$e^{4ka} = \frac{(1-\gamma)^2}{(1+\gamma)^2} = \left[\frac{2\frac{m\alpha}{k\hbar^2}}{2\left(1-\frac{m\alpha}{k\hbar^2}\right)}\right]^2 = \left[\frac{1}{\left(\frac{k\hbar^2}{m\alpha}-1\right)}\right]^2$$

let $ka \equiv z$, then

$$e^{-2z} = \pm\left(\frac{z\hbar^2}{m\alpha a}-1\right)$$

& let $c \equiv \frac{\hbar^2}{m\alpha a}$, so $\boxed{e^{-2z} = \pm(zc-1)}$



we will get 1 intersection for the line $zc-1$, always (for $c>0$).

The line $1-zc$ intersects as long as $c<2$, so there will be 2 bound states provided:

$$\frac{\hbar^2}{m\alpha a} < 2 \Rightarrow \frac{\hbar^2}{m\alpha} < 2a.$$

If $\alpha = \hbar^2/ma$, $c = \frac{\hbar^2}{ma\frac{\hbar^2}{ma}} = 1$, & $z_1 \approx 1.10886$ & $z_2 \approx .7968$ are the intersections.

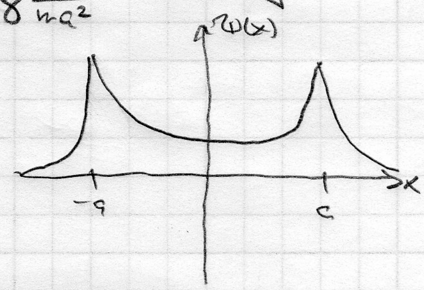
Then $z = ka = \sqrt{\frac{2mE}{\hbar^2}}\,a \Rightarrow -\frac{z^2}{a^2}\frac{\hbar^2}{2m} = E$ & for our two values, $E_1 \approx -.615\frac{\hbar^2}{ma^2}$

$$E_2 \approx -.317\frac{\hbar^2}{ma^2}$$

For $\alpha = \frac{\hbar^2}{4ma}$, $c = \frac{\hbar^2}{ma\frac{\hbar^2}{4ma}} = 4$ & $z_1 \approx .369418$ is the only solution, the energy in this case is

$$E_1 \approx -.068\frac{\hbar^2}{ma^2}$$

The lower (absolute) energy state is symmetric:



and the higher (absolute) energy state is antisymmetric