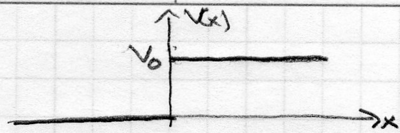
Problem 14.1

For $V(x) = \begin{cases} \emptyset & x < \emptyset \\ V_0 & x \geq \emptyset \end{cases}$



we have two regions, $x \leq \emptyset$ & $x \geq \emptyset$, let

$$\psi(x) = \begin{cases} \psi_I(x) & x \leq \emptyset \\ \psi_{II}(x) & x > \emptyset \end{cases}$$

On the left, $-\frac{\hbar^2}{2m}\psi_I'' = E\psi_I$, & on the right: $-\frac{\hbar^2}{2m}\psi_{II}'' + V_0\psi_{II} = E\psi_{II}$.

If $E = V_0$, we have: $\frac{-\hbar^2}{2m}\psi_{II}'' = \emptyset \Rightarrow \psi_{II}(x) = Ax + B$.
and on the left:

$$\psi_I'' = -\frac{2mV_0}{\hbar^2}\psi_I \quad , \text{let } k^2 \equiv \frac{2mV_0}{\hbar^2} \text{, then:}$$

$$\psi_I(x) = Ce^{ikx} + De^{-ikx}$$

Now we impose continuity at $x = \emptyset$: $\psi_I(\emptyset) = \psi_{II}(\emptyset)$

$$C + D = B \quad (*)$$

& derivative continuity gives: $\frac{d\psi_{II}}{dx}\Big|_\emptyset = \frac{d\psi_I}{dx}\Big|_\emptyset$

$$A = ik(C - D) \quad (+)$$

If we take $\psi_{II}(x) = B$ (set $A=\emptyset$) to avoid the linearly growing solution, then $(*)$ & $(+)$ give:

$$\left.\begin{matrix} C + D = B \\ C - D = \emptyset \end{matrix}\right\} \quad B = 2C.$$

our solutions read:

$$\boxed{\underset{(x \geq \emptyset)}{\psi_I(x) = 2C} \qquad \underset{(x \leq \emptyset)}{\psi_{II}(x) = 2C\cos\left(\sqrt{\frac{2mV_0}{\hbar^2}}\,x\right)}}$$

Problem 14.2

We know, from Ehrenfest's theorem, that

$$\frac{d\langle p\rangle}{dt} = \left\langle \frac{-dV}{dx} \right\rangle$$

& for a free particle, $V = \emptyset$, so $\langle -\frac{dV}{dx}\rangle = \emptyset$, & then $\frac{d\langle p\rangle}{dt} = \emptyset \Rightarrow \langle p\rangle = \text{const}$
so the initial value of $\langle p\rangle$ is preserved for all time.

Problem 14.3

Given $\overline{\Psi}(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$ (normalized).

The probability we make a measurement $E = -\frac{m\alpha^2}{2\hbar^2}$ is given by $|c_-|^2$ where

$$c_- = \int_{-\infty}^{+\infty} \Psi_-(x)^* \, \overline{\Psi}(x)\, dx$$

and $\Psi_-(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$ is the normalized delta-well-bound state.

$$= \left(\frac{2a}{\pi}\right)^{1/4} \frac{\sqrt{m\alpha}}{\hbar}\left[\underbrace{\int_{-\infty}^{0} e^{-(ax^2 - m\alpha x/\hbar^2)}dx + \int_0^\infty e^{-(ax^2 + m\alpha x/\hbar^2)}dx}_{\text{integrals are equal}}\right]$$

$$= 2\left(\frac{2a}{\pi}\right)^{1/4} \frac{\sqrt{m\alpha}}{\hbar}\left[\int_0^\infty e^{-(ax^2 + m\alpha x/\hbar^2)}dx\right]$$

let $y = \sqrt{a}\,x + F \Rightarrow y^2 - F^2 = ax^2 + \underbrace{2\sqrt{a}F}_{= \frac{m\alpha}{\hbar^2}}x \Rightarrow F = \frac{m\alpha}{2\sqrt{a}\hbar^2}$

& $dx = \frac{1}{\sqrt{a}}dy$, so:

$$= 2\left(\frac{2a}{\pi}\right)^{1/4} \frac{\sqrt{m\alpha}}{\hbar}\frac{1}{\sqrt{a}}\left[\int_{\frac{m\alpha}{2\sqrt{a}\hbar^2}}^{\infty} e^{-y^2}dy\right] e^{+\frac{m^2\alpha^2}{4a\hbar^4}}$$

The error function is: $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy$, so $\int_0^\infty e^{-y^2} = \underbrace{\int_0^z e^{-y^2}dy}_{\frac{\sqrt{\pi}}{2}\mathrm{erf}(z)} + \int_z^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}$

& $\int_z^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}(1 - \mathrm{erf}(z))$.

$$= 2\left(\frac{2a}{\pi}\right)^{1/4} \frac{\sqrt{m\alpha}}{\hbar}\frac{1}{\sqrt{a}} e^{+\frac{m^2\alpha^2}{4a\hbar^4}} \cdot \frac{\sqrt{\pi}}{2}\left(1 - \mathrm{erf}\left(\frac{m\alpha}{2\sqrt{a}\hbar^2}\right)\right)$$

let $z = \frac{m\alpha}{2\sqrt{a}\hbar^2}$, then

$$c_- = \left(2\pi \frac{4m^2\alpha^2}{4a\hbar^4}\right)^{1/4} e^{z^2}(1 - \mathrm{erf}(z))$$

$$= (8\pi \cdot z^2)^{1/4} e^{z^2}(1 - \mathrm{erf}(z)).$$

&

$$c_-^2 = (8\pi)^{1/2} z e^{2z^2}(1 - \mathrm{erf}(z))^2$$

$$= \boxed{2\sqrt{2\pi}\, z e^{2z^2}(1 - \mathrm{erf}(z))^2}$$

TOPS. 35502