Problem 12.1

a. For $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(k)\,\psi_k(x) e^{-iE(k)t/\hbar}\,dk$ w/ $\psi_k(x) = e^{ikx}$, $E(k) = \frac{\hbar^2 k^2}{2m}$

we have:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(k)\, e^{ik(x - \frac{\hbar k}{2m}t)}\,dk$$

& then:

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (ik)^2\Phi(k) e^{ik(x-\frac{\hbar k}{2m}t)}\,dk$$

&

$$\frac{\partial\Psi}{\partial t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\frac{-i\hbar k^2}{2m}\right)\Phi(k) e^{ik(x-\frac{\hbar k}{2m}t)}\,dk$$

& we can write:

$$-\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} k^2\Phi(k) e^{ik(x-\frac{\hbar k}{2m}t)}\,dk = -\frac{2m}{i\hbar}\frac{\partial\Psi}{\partial t}$$

or $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$ which is Schrödinger's eqn.

b. Starting w/ $\Psi(x,t) = \dfrac{\left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-amx^2}{m+2ia\hbar t}\right]}{(1/a + 2i\hbar t/m)^{1/2}} = \dfrac{\left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a + 2i\hbar t/m}\right]}{(1/a + 2i\hbar t/m)^{1/2}}$.

we have:

$$\frac{\partial\Psi}{\partial t} = \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{2i\hbar x^2/m}{(1/a+2i\hbar t/m)^2(1/a+2i\hbar t/m)^{1/2}} - \frac{i\hbar/m}{(1/a+2i\hbar t/m)^{3/2}}\right]$$

$$= \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{2i\hbar x^2/m - i\hbar/m(1/a+2i\hbar t/m)}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

$$= \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{i\hbar/m\,(2x^2 - 1/a - 2i\hbar t/m)}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

&

$$\frac{\partial\Psi}{\partial x} = \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{-2x}{(1/a+2i\hbar t/m)^{3/2}}\right]$$

then

$$\frac{\partial^2\Psi}{\partial x^2} = \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{-2}{(1/a+2i\hbar t/m)^{3/2}} + \frac{4x^2}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

$$= \left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{2(2x^2 - 1/a - 2i\hbar t/m)}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

Now we construct $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$ & $i\hbar\frac{\partial\Psi}{\partial t}$ & see if they are equal:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{\hbar^2}{m}\left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{2x^2 - 1/a - 2i\hbar t/m}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{m}\left(\frac{2}{\pi a}\right)^{1/4}\exp\left[\frac{-x^2}{1/a+2i\hbar t/m}\right]\left[\frac{2x^2 - 1/a - 2i\hbar t/m}{(1/a+2i\hbar t/m)^{5/2}}\right]$$

equal so in this case:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$$

Problem 12.2

a. For $\overline{\Psi}(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$ we know that: $\int_{-\infty}^{+\infty} \overline{\Psi}^*(x)\overline{\Psi}(x)dx = 1$.
& the initial probability density is:

$$\overline{\rho}(x) = \overline{\Psi}(x)^* \overline{\Psi}(x) = \left(\frac{2a}{\pi}\right)^{1/2} e^{-2ax^2}$$

If we take: $\tilde{\Psi}(x) = e^{if(x)}\overline{\Psi}(x)$, then the initial probability density is:

$$\tilde{\rho}(x) = \tilde{\Psi}(x)^* \tilde{\Psi}(x) = e^{-if(x)}\overline{\Psi}(x)\, e^{if(x)}\overline{\Psi}(x) = \overline{\Psi}(x)^* \overline{\Psi}(x) = \overline{\rho}(x).$$

b. For $\tilde{\Psi}(x) = e^{if(x)}\overline{\Psi}(x)$, we have:

$$\langle p \rangle = \int_{-\infty}^{+\infty} \tilde{\Psi}^*(x) \cdot \frac{\hbar}{i}\frac{\partial \tilde{\Psi}(x)}{\partial x} dx = \frac{\hbar}{i}\int_{-\infty}^{+\infty} e^{-if(x)}\overline{\Psi}(x)\left[+if'(x)\overline{\Psi}(x) + \overline{\Psi}'(x)\right]e^{if(x)} dx$$

$$= \frac{\hbar}{i}\int_{-\infty}^{+\infty} \overline{\Psi}(x)\cdot\overline{\Psi}'(x)dx + \hbar\int_{-\infty}^{+\infty} f'(x)\overline{\Psi}(x)\overline{\Psi}(x)dx$$

we know this is zero.

$$= \hbar\int_{-\infty}^{+\infty} f'(x)\overline{\Psi}(x)\overline{\Psi}(x)dx$$

the simplest $f(x)$, then, is $\boxed{f(x) = \beta x}$, then $f'(x)$ is a constant, &

$$= \beta\hbar.$$

Problem 12.3

We know that $\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$, then the dimension of $\delta(x)$ is given by:

$$|f|\,|\delta|\,|dx| = |f| \Rightarrow |\delta| = 1/|dx|$$

& $|dx|$ is a length, so $|\delta| = 1/L$, if x is in meters, $\boxed{\delta(x) \text{ is in } 1/m}$

(then $\rho = \delta(x - x(t))$ has the correct units as a probability density).