Problem 11.1

a. We want the probability of finding a "classical" particle between $-\frac{1}{\sqrt{2}}x_0$ & $\frac{1}{\sqrt{2}}x_0$ (i.e. $\pm\sigma$) of zero. That means we need to compute:

$$P = \int_{-x_0/\sqrt{2}}^{x_0/\sqrt{2}} \rho(x)dx = \int_{-x_0/\sqrt{2}}^{x_0/\sqrt{2}} \frac{1}{\pi}\frac{1}{\sqrt{x_0^2-x^2}}dx$$

let $x = x_0\sin\theta \Rightarrow dx = x_0\cos\theta d\theta$ & $\sqrt{x_0^2-x^2} = x_0\cos\theta$

$$= \int_{-\pi/4}^{+\pi/4}\frac{1}{\pi}d\theta = \frac{1}{\pi}\theta\Big|_{\theta=-\pi/4}^{+\pi/4} = \boxed{1/2}$$

b. The probability of finding the particle between $9/10\, x_0$ & x_0 is:

$$P = \int_{9/10 x_0}^{x_0}\rho(x)\,dx = \int_{9/10 x_0}^{x_0}\frac{1}{\pi}\frac{1}{\sqrt{x_0^2-x^2}}dx = \frac{1}{\pi}\int_{\sin^{-1}(9/10)}^{\pi/2}d\theta = \frac{1}{\pi}\left[\pi/2 - \sin^{-1}(9/10)\right] \approx \boxed{.14}$$

Problem 11.2

We have $f(x) = f_0\cos(qx)$, & the Fourier Transform is defined to be

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_0\cos(qx)e^{-ikx}dx$$

using: $\cos(qx) = \frac{1}{2}\left[e^{iqx}+e^{-iqx}\right]$, we can write:

$$= \frac{f_0}{2\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty}e^{i(q-k)x}dx + \int_{-\infty}^{+\infty}e^{-i(q+k)x}dx\right]$$

& we know that $\int_{-\infty}^{+\infty}e^{is(x-a)}ds = 2\pi\,\delta(x-a)$, so:

$$= \frac{1}{2}f_0\sqrt{2\pi}\left[\delta(q-k)+\delta(q+k)\right]$$

or

$$\boxed{\tilde{f}(k) = \sqrt{\frac{\pi}{2}}\,f_0\left[\delta(q-k)+\delta(q+k)\right].}$$

Problem 11.3 (Griffiths 2.21)

b. We are given: $\Psi(x,0) = Ae^{-a|x|} \equiv \Psi(x)$

The time-dependent solution is:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(k) e^{+ikx} e^{-i\frac{\hbar k^2}{2m}t}\, dk$$

w/

$$\begin{aligned}
\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Psi(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Ae^{-a|x|} e^{-ikx}\, dx \\
&= \frac{A}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-ax-ikx}\, dx + \int_{-\infty}^{0} e^{ax-ikx}\, dx\right] \\
&= \frac{A}{\sqrt{2\pi}}\left[\left.\frac{-e^{-ax-ikx}}{(a+ik)}\right|_{x=0}^{\infty} + \left.\frac{e^{ax-ikx}}{(a-ik)}\right|_{x=-\infty}^{0}\right] \\
&= \frac{A}{\sqrt{2\pi}}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \frac{A}{\sqrt{2\pi}}\left[\frac{2a}{a^2+k^2}\right] = \boxed{\sqrt{\frac{2a}{\pi}}\cdot\frac{a}{a^2+k^2}}
\end{aligned}$$

a. $$\int_{-\infty}^{+\infty} A^2 e^{-2a|x|}\, dx = 2A^2\int_0^{\infty} e^{-2ax}\, dx = -\frac{2A^2}{2a} e^{-2ax}\Big|_{x=0}^{\infty} = A^2/a = 1 \Rightarrow A = \boxed{\sqrt{a}}$$

c. $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(k) e^{ikx} e^{-i\hbar k^2/2m\, t}\, dk = \boxed{\frac{a^{3/2}}{\pi}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{a^2+k^2} e^{-i\hbar k^2/2m\, t}\, dk}$$

d. For a large, $\Psi(x,0)$ is sharply peaked, for a small, $\Psi(x,0)$ is broad. The spread of $\Phi(k)$ follows the opposite pattern. Thus for a sharply peaked position density, there is a wide range of k (& hence momenta, since $p = \hbar k$ for the individual "stationary" states.